

On some developments in the Nonsymmetric Kaluza–Klein Theory

M. W. Kalinowski

Faculty of Physics, Warsaw University, ul. Hoża 69, 00-681 Warsaw, Poland;
Pracownia Bioinformatyki, Instytut Medycyny Doświadczalnej i Klinicznej PAN,
ul. Pawińskiego 5, 02-106 Warszawa, Poland
e-mail: markwkal@bioexploratorium.pl, mkalinowski@imdik.pan.pl

Abstract

We consider a condition for a charge confinement and gravito-electromagnetic wave solutions in the Nonsymmetric Kaluza–Klein Theory. We consider also an influence of a cosmological constant on a static, spherically symmetric solution. We remind to the reader some fundamentals of the Nonsymmetric Kaluza–Klein Theory and a geometrical background behind the theory. Simultaneously we give some remarks concerning misunderstanding connected to several notions of Kaluza–Klein Theory, Einstein Unified Field Theory, geometrization and unification of physical interactions. We reconsider Dirac field in the Nonsymmetric Kaluza–Klein Theory.

Introduction

The Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory has been developed (see Refs [1], [2], [3], [4]). The theory unifies gravitational theory described by NGT (Nonsymmetric Gravitational Theory) (see Ref. [5]) and Electrodynamics. The theory has been extended to Nonabelian Gauge Fields, Higgs Fields, scalar field with applications to cosmology. Some possibilities to get a confinement of colour have been suggested. A nonsingular spherically symmetric solution has been derived. The Nonsymmetric Kaluza–Klein Theory can be obtained from the Nonsymmetric Jordan–Thiry Theory by putting the scalar field into zero. In this way it is a limit of the Nonsymmetric Jordan–Thiry Theory. The Nonsymmetric Jordan–Thiry Theory has several physical applications in cosmology, e.g.: 1. cosmological constant, 2. inflation, 3. quintessence, and some possible relations to a dark matter problem. Simultaneously the theory unifies gravity with gauge fields in a nontrivial way via geometrical unification of two fundamental invariance principles in physics: 1. coordinate invariance principle, 2. gauge invariance principle. Unification on the level of invariance principles is more important than on a level of interactions for from invariance principles we get conservation laws (via Noether theorem). In some sense Kaluza–Klein theory unifies an energy-momentum conservation law with an electric charge conservation law. This unification has been achieved in more than 4-dimensional world. It is nontrivial for we can get some additional effects unknown in conventional theories of gravity and gauge fields (electromagnetic or Yang–Mills fields). All of these effects which we call “interference effects” between gravity

and gauge fields are testable in principle in an experiment or an observation. The formalism of this unification has been described in references [1]–[4]. The Nonsymmetric Kaluza–Klein Theory is an example of a geometrization of gravitational and electromagnetic interactions according to the Einstein programme. In this paper we consider conditions for a charge confinement in the theory and three solutions of Nonsymmetric Kaluza–Klein Theory equations describing gravito–electromagnetic waves. We consider also an influence of a cosmological constant on a static, spherically symmetric solution. This solution can be considered as a model of an electron for it has remarkable properties being nonsingular in electric and gravitational fields. Simultaneously this solution has been built from elementary fields. The properties of the solution can be considered as “interference effects” between electromagnetic and gravitational fields in our unification. In this way the theory achieves an old dream of Einstein, Weyl, Kaluza, Eddington and Schrödinger on a **unitary classical field theory** by having particles as spherically symmetric singularity-free solutions of the field equations.

The Nonsymmetric Kaluza–Klein Theory should be called a Unified Field Theory according to the definition which we quote here (see Ref. [6]): “*Unified Field Theory: Any theory which attempts to express gravitational theory and electromagnetic theory within a single unified framework. Usually, an attempt to generalize Einstein’s general theory of relativity from a theory of gravitation alone to a theory of gravity and classical electromagnetism*”. In our case this single unified framework is a multidimensional analogue of geometry from Einstein Unified Field Theory (treated as a generalized gravity) defined on an electromagnetic bundle.

Summing up the Nonsymmetric Kaluza–Klein Theory connects old ideas of unitary field theories (unified field theories) with some modern applications.

The paper has been divided into four sections. In the first section we give some elements of the Nonsymmetric Kaluza–Klein Theory in some new setting. We give also a condition for a dielectric confinement of a charge. In the second section we give three solutions of the field equations describing gravito–electromagnetic waves. In the third section we deal with a spherically-symmetric solution in a presence of a cosmological constant. In the fourth section we give a theory of a Dirac field in the Nonsymmetric Kaluza–Klein Theory getting CP-violation and EDM (Electric Dipole Moment) for a fermion. We reconsider some notion known from our previous papers. In Conclusions we give also some remarks concerning some misunderstanding concerning Kaluza–Klein Theory, Einstein Unified Field Theory commonly met. We put our investigations on a wider background. In Appendix A we give some notions of differential geometry used in the paper and in Appendix B some details of calculations. In Appendix C we give some elements of Clifford algebra and spinor theory. Appendix D is devoted to a redefinition of the Nonsymmetric Kaluza–Klein Theory in terms of GR (General Relativity) and additional “matter fields”.

In this paper we use the following convention. Capital Latin indices $A, B, C = 1, 2, 3, 4, 5$ (Kaluza–Klein indices), lower Greek cases $\alpha, \beta, \gamma = 1, 2, 3, 4$ (space-time indices), lower Latin cases $a, b, c = 1, 2, 3$ (space indices). In Appendix A we use Latin lower indices in a Lie algebra \mathfrak{G} of a Lie group G , $a, b, c = 1, 2, \dots, n = \dim \mathfrak{G} = \dim G$. This cannot cause any misunderstanding. In Appendix B we use capital Latin indices $A, B, C, W, N, M = 1, 2, \dots, n$, where n is the dimension of a manifold equipped with nonsymmetric tensor and a nonsymmetric connection. This also does not result in any misunderstanding.

1 Elements of the Nonsymmetric Kaluza–Klein Theory

The basic logic of a construction is as follows. We define a Nonsymmetric Kaluza–Klein Theory as a five-dimensional analogue of NGT using our extension of natural metrization of an electromagnetic fibre bundle achieving in this way a unification of two fundamental principles of invariance (i.e. a coordinate invariance principle and a gauge invariance principle) reducing both to the coordinate invariance principle in 5-dimensional world (see Ref. [4] for details).

Let us notice that our construction from Ref. [4] is more general for it contains a scalar field ρ (or Ψ) which here is put to $\rho = 1$ ($\Psi = 0$).

Let \underline{P} be a principal fibre bundle with a structural group $G = U(1)$ over a space-time E with a projection π and let us define on this bundle a connection α . We call this bundle an *electromagnetic bundle* and α an *electromagnetic connection* (see Appendix A for details). We define a curvature 2-form for the connection α

$$\Omega = d\alpha \equiv \frac{1}{2} \pi^* (F_{\mu\nu} \bar{\theta}^\mu \wedge \bar{\theta}^\nu), \quad \mu, \nu = 1, 2, 3, 4, \quad (1.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad e^* \alpha = A_\mu \bar{\theta}^\mu. \quad (1.2)$$

A_μ is a 4-potential of the electromagnetic field, e is a local section of $\underline{P}(e : E \supset U \rightarrow P)$, $F_{\mu\nu}$ is an electromagnetic field strength, and $\bar{\theta}^\mu$ is a frame on E . Bianchi identity is

$$d\Omega = 0, \quad (1.3)$$

so the 4-potential exists. This is of course simply the first pair of Maxwell equations. On the space-time E we define a nonsymmetric metric tensor $g_{\alpha\beta}$ such that

$$\begin{aligned} g_{\alpha\beta} &= g_{(\alpha\beta)} + g_{[\alpha\beta]} \\ g_{\alpha\beta} g^{\gamma\beta} &= g_{\beta\alpha} g^{\beta\gamma} = \delta_\alpha^\gamma, \end{aligned} \quad (1.4)$$

where the order of indices is important. In such a way we suppose that

$$g = \det g_{\alpha\beta} \neq 0. \quad (1.5)$$

We suppose also that

$$\tilde{g} = \det g_{(\alpha\beta)} \neq 0, \quad (1.6)$$

defining an inverse tensor $\tilde{g}^{(\alpha\beta)}$ for $g_{(\alpha\beta)}$ such that $\tilde{g}^{(\alpha\beta)} g_{(\beta\mu)} = \delta_\mu^\alpha$. The combination of symmetric and antisymmetric tensor Eq.(1.4) is going to new inside in an inverse tensor. We define also on E two connections \bar{w}^α_β and \bar{W}^α_β

$$\bar{w}^\alpha_\beta = \bar{\Gamma}^\alpha_{\beta\gamma} \bar{\theta}^\gamma \quad (1.7)$$

and

$$\bar{W}^\alpha_\beta = \bar{W}^\alpha_{\beta\gamma} \bar{\theta}^\gamma, \quad \alpha, \beta, \gamma = 1, 2, 3, 4, \quad (1.8)$$

such that

$$\bar{W}^\alpha_\beta = \bar{w}^\alpha_\beta - \frac{2}{3} \delta^\alpha_\beta \bar{W}, \quad (1.9)$$

where

$$\overline{W} = \overline{W}_\gamma \overline{\theta}^\gamma = \frac{1}{2}(\overline{W}^\sigma_{\gamma\sigma} - \overline{W}^\sigma_{\sigma\gamma})\overline{\theta}^\gamma.$$

For the connection $\overline{w}^\alpha_\beta$ we suppose the following conditions

$$\begin{aligned} \overline{D}g_{\alpha+\beta-} &= \overline{D}g_{\alpha\beta} - g_{\alpha\delta}\overline{Q}^\delta_{\beta\gamma}(\overline{\Gamma})\overline{\theta}^\gamma = 0 \\ \overline{Q}^\alpha_{\beta\alpha}(\overline{\Gamma}) &= 0, \end{aligned} \quad (1.10)$$

where \overline{D} is the exterior covariant derivative with respect to $\overline{w}^\alpha_\beta$ and $\overline{Q}^\alpha_{\beta\alpha}(\overline{\Gamma})$ is a torsion of $\overline{w}^\alpha_\beta$. $\overline{W}^\alpha_\beta$ is called an unconstrained connection and $\overline{w}^\alpha_\beta$ a constrained connection. Thus we have defined on space-time all quantities present in Moffat's theory of gravitation (NGT, see Ref. [7]). In this approach we consider test particles moving along geodesics with respect to a Levi-Civita connection generated by a tensor $g_{(\alpha\beta)}$ on E , i.e. $\tilde{\overline{w}}^\alpha_\beta = \tilde{\overline{\Gamma}}^\alpha_{\beta\gamma}\overline{\theta}^\gamma$. Let us introduce on P a frame (a lift horizontal base)

$$\theta^A = (\pi^*(\overline{\theta}^\alpha), \lambda\alpha = \theta^5), \quad \lambda = \text{const.} \quad (1.11)$$

Now we turn to the natural nonsymmetric metrization of the bundle P . We have

$$\overline{\gamma} = \pi^*\overline{g} - \theta^5 \otimes \theta^5 = \pi^*(g_{(\alpha\beta)}\overline{\theta}^\alpha \otimes \overline{\theta}^\beta) - \theta^5 \otimes \theta^5 \quad (1.12)$$

$$\underline{\gamma} = \pi^*\underline{g} = \pi^*(g_{[\alpha\beta]}\overline{\theta}^\alpha \wedge \overline{\theta}^\beta) \quad (1.13)$$

where \overline{g} is a symmetric tensor on E , $\overline{g} = g_{(\alpha\beta)}\overline{\theta}^\alpha \otimes \overline{\theta}^\beta$ and \underline{g} is a 2-form on E , $\underline{g} = g_{[\alpha\beta]}\overline{\theta}^\alpha \wedge \overline{\theta}^\beta$. Taking both parts together we get

$$\gamma = \pi^*g - \theta^5 \otimes \theta^5 = \pi^*(g_{\alpha\beta}\overline{\theta}^\alpha \otimes \overline{\theta}^\beta) - \theta^5 \otimes \theta^5. \quad (1.14)$$

The nonsymmetric metric γ is biinvariant with respect to the action of a group $U(1)$ on P . From the classical Kaluza-Klein theory we know that $\lambda = 2\frac{\sqrt{G_N}}{c^2}$. We work with such a system of units that $G_N = c = 1$ and $\lambda = 2$. Thus we have in a matrix form

$$\begin{aligned} \gamma_{AB} &= \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & -1 \end{array} \right), \\ \gamma &= \gamma_{AB}\theta^A \otimes \theta^B. \end{aligned} \quad (1.15)$$

The tensor γ_{AB} has this shape in a lift horizontal base, which is of course nonholonomic ($d\theta^5 \neq 0$). We can find it in a holonomic system of coordinates. Let us take a section $e : E \supset U \rightarrow P$ and attach to it a coordinate x^5 , selecting $x^\mu = \text{const}$ on the fibre in such a way that e is given by the condition $x^5 = 0$ and $\zeta_5 = \partial/\partial x^5$. Then we have $e^*dx^5 = 0$ and

$$\alpha = \frac{1}{\lambda}dx^5 + \pi^*(A_\mu\overline{\theta}^\mu), \quad \text{where} \quad A = A_\mu\overline{\theta}^\mu = e^*\alpha.$$

Taking $\overline{\theta}^\mu = dx^\mu$ one gets $\theta^5 = dx^5 + \pi^*(\lambda A_\mu dx^\mu)$. Putting the last result into Eq (1.14) one finds

$$\begin{aligned} \gamma &= \pi^*\left((g_{\alpha\beta} - \lambda^2 A_\alpha A_\beta) dx^\alpha \otimes dx^\beta\right) \\ &\quad - \pi^*(\lambda A_\alpha dx^\alpha) \otimes dx^5 - dx^5 \otimes (\lambda A_\beta dx^\beta) - dx^5 \otimes dx^5. \end{aligned} \quad (1.16)$$

In this coordinate system the tensor γ takes a matrix form

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} - \lambda^2 A_\alpha A_\beta & -\lambda A_\alpha \\ \hline -\lambda A_\beta & -1 \end{array} \right). \quad (1.17)$$

In order to have the correct dimension of a four-potential we should rather write $e^* \alpha = (q/\hbar c) A = \mu A$, where q is an elementary charge and \hbar is Planck's constant. The same is true for the curvature of connection on the electromagnetic bundle $\Omega = \lambda \mu \pi^*(F)$, $F = \frac{1}{2} F_{\mu\nu} \bar{\theta}^\mu \bar{\theta}^\nu$. Moreover, it can be absorbed by a constant λ (we have only one constant as in classical Kaluza theory and a Planck's constant appearance is illusory for the theory is classical (not quantum) and eventually demands quantization).

In this way Eq. (1.17) gives us a classical Kaluza–Klein approach with a five-dimensional metric tensor and with a Killing vector ζ_5 . Even if in Eq. (1.15) we have not any four-potential A_μ , an electromagnetic field exists as an electromagnetic connection α .

The connection contains (potentially) all possible four-potential A_μ (it means, in any possible gauge). To choose a gauge means here to take a section of a bundle. An electromagnetic connection α is really an electromagnetic field. We can also consider $e^* \Omega$. Moreover, the structural group of an electromagnetic bundle $U(1)$ is abelian. It means $\Omega = d\alpha + \frac{1}{2}[\alpha, \alpha] = d\alpha$. This means we can use Eq. (1.1). In this theory we have two more fibre bundles. Two fibre bundles of frames over E (a space-time) and over P (a bundle manifold). Moreover, in order to simplify a formalism we do not refer explicitly to those fibre bundles.

Tensor (1.15) in a lift horizontal base looks simpler than (1.17), moreover, we do not lose any information. Introducing nonholonomic frames we can write very complicated tensors as diagonal and sometimes it causes some misunderstanding. In this way $F_{\mu\nu}$ tensor is connected to A_μ —four-potential via a curvature of an electromagnetic fibre bundle and not via a metric (1.17) which has more mathematical sound. Eq. (1.13) is equivalent to Eq. (1.17). The first one is written in a lift horizontal frame and the second in $dx^A = (dx^\alpha, dx^5)$.

Now we define on P a connection w^A_B such that

$$\begin{aligned} w^A_B &= \Gamma^A_{BC} \theta^C, \quad A, B, C = 1, 2, 3, 4, 5, \\ D\gamma_{A+B-} &= D\gamma_{AB} - \gamma_{AD} Q^D_{BC}(\Gamma) \theta^C = 0, \end{aligned} \quad (1.18)$$

which is invariant with respect to an action of the group $U(1)$ on P . D is an exterior covariant derivative with respect to the connection w^A_B and $Q^D_{BC}(\Gamma)$ is its torsion. Let us notice that for w^A_B we do not suppose any constraints on its torsion. In Refs [1], [2], [3] it is shown that

$$w^A_B = \left(\begin{array}{c|c} \pi^*(\bar{w}^\alpha_\beta) + g^{\gamma\alpha} H_{\gamma\beta} \theta^5 & H_{\beta\gamma} \theta^\gamma \\ \hline g^{\alpha\beta} (H_{\gamma\beta} + 2F_{\beta\gamma}) \theta^\gamma & 0 \end{array} \right) \quad (1.19)$$

where $H_{\beta\gamma}$ is a tensor on E such that

$$g_{\delta\beta} g^{\gamma\delta} H_{\gamma\alpha} + g_{\alpha\delta} g^{\delta\gamma} H_{\beta\gamma} = 2g_{\alpha\delta} g^{\delta\gamma} F_{\beta\gamma}. \quad (1.20)$$

It is possible to prove that (see Appendix B)

$$H_{\gamma\beta} = -H_{\beta\gamma} \quad (\text{if } F_{\mu\nu} = -F_{\nu\mu}). \quad (1.21)$$

Tensors $H_{\mu\nu}$ and $F_{\mu\nu}$ define a five-dimensional connection on P . In the case of a symmetric tensor $g_{\alpha\beta}$, $H_{\mu\nu} = F_{\mu\nu}$ and the theory reduces to ordinary Kaluza–Klein theory.

We define on P a second connection

$$\begin{aligned} W^A_B &= w^A_B - \frac{4}{9}\delta^A_B \overline{W} \\ \overline{W} &= \text{hor } \overline{W}. \end{aligned} \quad (1.22)$$

Connection W^A_B is a five-dimensional analogue of the connection $\overline{W}^\alpha_\beta$ known in Einstein Unified Field Theory and NGT (Moffat theory of gravitation, see Ref. [7]). According to our notation “ $\overline{}$ ” over a symbol means the quantity is defined on E (a space-time), “ \sim ” over a symbol means the quantity is defined with respect to Levi–Civita connections, i.e. $\tilde{T}^\alpha_{\beta\gamma}$ mean coefficients of Levi–Civita connections on E .

The connections (1.19) and (1.22) unify electromagnetic and gravitational interactions in the Nonsymmetric Kaluza–Klein Theory. In the theory we can also consider a dual frame $\zeta_A = (\zeta_\alpha, \zeta_5)$ such that $\theta^A(\zeta_B) = \delta^A_B$. In this way $[\zeta_\alpha, \zeta_\beta] = \frac{\lambda}{2}F_{\alpha\beta}\zeta_5$ and the remaining commutators of vector fields vanish. In the classical Kaluza–Klein Theory the geodetic equations describe a motion of a charged particle (test particle), i.e. we get a Lorentz force term. Moreover, in the case of classical (Riemannian) Kaluza–Klein Theory we have to do with only one (Levi–Civita) connection on P . Here we have to do with several possibilities (see Ref. [4]). For we have $H_{\mu\nu} = -H_{\nu\mu}$ it seems now that we should choose a Riemannian part of (1.19). It means Levi–Civita connection generated by $\gamma_{(AB)}$.

This means we have $u^B \tilde{\nabla}_B u^A = 0$, where $\tilde{\nabla}$ means a covariant derivative with respect to $\tilde{w}^A_B = \tilde{T}^A_{BC}\theta^C$ (a Riemannian part of a connection w^A_B), $u^A(\tau)$ is a tangent vector to a geodetic line. Eventually one gets

$$\frac{\tilde{D}u^\alpha}{d\tau} + \frac{q}{m_0} \tilde{g}^{(\alpha\mu)} F_{\mu\beta} u^\beta = 0, \quad \frac{q}{m_0} = 2u^5 = \text{const.} \quad (1.23)$$

$2u^5$ has an interpretation as q/m_0 for a test particle, where q is a charge and m_0 is a rest mass of a test particle. $\frac{\tilde{D}}{d\tau}$ means a covariant derivative with respect to \tilde{w}^α_β along a curve to which $u(\tau)$ is tangent. $\tilde{w}^\alpha_\beta = \tilde{T}^\alpha_{\beta\gamma}\tilde{\theta}^\gamma$ is of course a Riemannian part of $\overline{w}^\alpha_\beta = \overline{T}^\alpha_{\beta\gamma}\overline{\theta}^\gamma$ (a Levi–Civita connection generated by $g_{(\alpha\beta)}$).

Let us calculate a Moffat–Ricci curvature scalar for W^A_B , $R(W)$, $R(W)\sqrt{\det \gamma_{AB}}$ is a five-dimensional Lagrangian density. One gets

$$R(W) = \overline{R}(\overline{W}) + (2(g^{[\mu\nu]}F_{\mu\nu})^2 - H^{\mu\nu}F_{\mu\nu}) \quad (1.24)$$

where

$$\overline{R}(\overline{W}) = g^{\mu\nu}\overline{R}_{\mu\nu}(\overline{T}) + \frac{2}{3}g^{[\mu\nu]}\overline{W}_{[\mu,\nu]} \quad (1.25)$$

is a Moffat–Ricci scalar for the connection $\overline{W}^\alpha_\beta$ and $\overline{R}_{\alpha\beta}(\overline{T})$ is a Moffat–Ricci tensor for the connection $\overline{w}^\alpha_\beta$. In particular

$$\overline{R}_{\mu\nu}(\overline{T}) = \overline{R}^\alpha_{\mu\nu\alpha}(\overline{T}) + \frac{1}{2}\overline{R}^\alpha_{\alpha\mu\nu}(\overline{T}), \quad (1.26)$$

where $\overline{R}^\alpha_{\mu\nu\rho}(\overline{\Gamma})$ are components of the ordinary curvature tensor for $\overline{\Gamma}$. In addition

$$H^{\mu\alpha} = g^{\beta\mu} g^{\gamma\alpha} H_{\beta\gamma}. \quad (1.27)$$

The action of the theory simply reads

$$S = \int_V \sqrt{\det \gamma_{AB}} R(W) d^5x.$$

Using Palatini variation principle with respect to $g_{\alpha\beta}$, A_μ , $\overline{W}^\alpha_\beta$, i.e.

$$0 = \delta \int_V \sqrt{\det \gamma_{AB}} R(W) d^5x = 2\pi\delta \int_U \sqrt{-g} \left(\overline{R}(\overline{W}) + (2(g^{[\mu\nu]} F_{\mu\nu})^2 - H^{\mu\nu} F_{\mu\nu}) \right) d^4x \quad (1.28)$$

(Eqs (1.24) and (1.28) give us 5-dimensional action in terms of 4-dimensional quantities which can be compared to the standard field theory action), where $V = U \times U(1)$, $U \subset E$, one gets from (1.28) fields equations

$$\overline{R}_{\alpha\beta}(\overline{W}) - \frac{1}{2} g_{\alpha\beta} \overline{R}(\overline{W}) = 8\pi T^{\text{em}}_{\alpha\beta} \quad (1.29)$$

$$\mathcal{G}^{[\mu\nu]}_{,\nu} = 0 \quad (1.30)$$

$$g_{\mu\nu,\sigma} - g_{\zeta\nu} \overline{\Gamma}^\zeta_{\mu\sigma} - g_{\mu\zeta} \overline{\Gamma}^\zeta_{\sigma\nu} = 0 \quad (1.31)$$

$$\partial_\mu (\mathcal{H}^{\alpha\mu}) = 2\mathcal{G}^{[\alpha\beta]} \partial_\beta (g^{[\mu\nu]} F_{\mu\nu}) \quad (1.32)$$

where

$$T^{\text{em}}_{\alpha\beta} = \frac{1}{4\pi} \left\{ g_{\gamma\beta} g^{\tau\mu} g^{\varepsilon\gamma} H_{\mu\alpha} H_{\tau\varepsilon} - 2g^{[\mu\nu]} F_{\mu\nu} F_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} (H^{\mu\nu} F_{\mu\nu} - 2(g^{[\mu\nu]} F_{\mu\nu})^2) \right\} \quad (1.33)$$

$$\mathcal{G}^{[\mu\nu]} = \sqrt{-g} g^{[\mu\nu]} \quad (1.34)$$

$$\mathcal{H}^{\mu\nu} = \sqrt{-g} g^{\beta\mu} g^{\gamma\alpha} H_{\beta\gamma} = \sqrt{-g} H^{\mu\alpha} \quad (1.35)$$

One can prove

$$H^{\mu\nu} H_{\mu\nu} = F^{\mu\nu} H_{\mu\nu}, \quad (1.36)$$

$$g^{[\mu\nu]} F_{\mu\nu} = g^{[\mu\nu]} H_{\mu\nu} \quad (1.37)$$

and

$$g^{\sigma\nu} g^{\alpha\mu} H_{\sigma\alpha} F_{\mu\nu} + g^{\mu\sigma} g^{\nu\beta} H_{\beta\sigma} F_{\mu\nu} = 2g^{\mu\sigma} g^{\nu\beta} F_{\mu\nu} F_{\beta\sigma}.$$

We have also

$$g^{\alpha\beta} T^{\text{em}}_{\alpha\beta} = 0. \quad (1.38)$$

Equations (1.29)–(1.32) can be written in the form

$$\overline{R}_{(\alpha\beta)}(\overline{\Gamma}) = 8\pi T^{\text{em}}_{(\alpha\beta)} \quad (1.39)$$

$$\overline{R}_{[[\alpha\beta],\gamma]}(\overline{\Gamma}) = 8\pi T^{\text{em}}_{[[\alpha\beta],\gamma]} \quad (1.40)$$

$$\overline{\Gamma}_\mu = 0 \quad (1.41)$$

$$g_{\mu\nu,\sigma} - g_{\zeta\nu} \overline{\Gamma}^\zeta_{\mu\sigma} - g_{\mu\zeta} \overline{\Gamma}^\zeta_{\sigma\nu} = 0 \quad (1.42)$$

$$\partial_\mu (\mathcal{H}^{\alpha\mu} - 2\mathcal{G}^{[\alpha\mu]} (g^{[\nu\beta]} F_{\mu\nu})) = 0 \quad (1.43)$$

where $\overline{R}_{\alpha\beta}(\overline{T})$ is a Moffat–Ricci tensor for the connection $\overline{w}^\alpha_\beta = \overline{T}^\alpha_{\beta\gamma}\overline{\theta}^\gamma$ and

$$\overline{T}_\mu = \overline{T}^\alpha_{[\mu\alpha]}, \quad (1.44)$$

$_{,\gamma}$ means a partial derivative with respect to x^γ (as usual).

Four-dimensional quantities in the theory A_μ , $g_{(\mu\nu)}$ and $g_{[\mu\nu]}$ are an electromagnetic field, a metric and a skew-symmetric tensor. They correspond to particles: a photon (a spin one), a graviton (a spin 2) and a skewon (a spin zero).

In the theory we get a current density

$$\overline{J}^\alpha = 2\partial_\mu(\sqrt{-g}g^{[\alpha\mu]}(g^{[\nu\beta]}F_{\nu\beta})) \quad (1.45)$$

which is conserved by its definition (in this way it is a topological current). Equation (1.43) can be written in the form

$$\overline{\nabla}_\mu H^{\alpha\mu} = J^\alpha \quad (1.46)$$

$$J^\alpha = 2\overline{\nabla}_\mu(g^{[\alpha\mu]}(g^{[\nu\beta]}F_{\nu\beta})) = 2g^{[\alpha\beta]}\overline{\nabla}_\mu(g^{[\nu\beta]}F_{\nu\beta}) \quad (1.47)$$

where $\overline{\nabla}_\mu$ is a covariant derivative for the connection $\overline{w}^\alpha_\beta$.

Equation (1.20) can be solved with respect to $H_{\nu\mu}$ (see Appendix B)

$$H_{\nu\mu} = F_{\nu\mu} - \tilde{g}^{(\tau\alpha)}F_{\alpha\nu}g_{[\mu\tau]} + \tilde{g}^{(\tau\alpha)}F_{\alpha\mu}g_{[\nu\tau]}. \quad (1.48)$$

However the form of Eq. (1.20) is easier to handle from theoretical point of view. Writing $H_{\mu\nu}$ in the form

$$H_{\mu\nu} = F_{\mu\nu} - 4\pi M_{\mu\nu} \quad (1.49)$$

we get

$$Q^5_{\mu\nu} = 8\pi M_{\mu\nu} = 2\tilde{g}^{(\tau\alpha)}(F_{\alpha\mu}g_{[\nu\tau]} - F_{\alpha\nu}g_{[\mu\tau]}), \quad (1.50)$$

where $Q^5_{\mu\nu}$ is a torsion in the fifth dimension for the connection w^A_B on P and $M_{\mu\nu}$ is an electromagnetic polarization tensor induced by a nonsymmetric tensor $g_{\alpha\beta}$ (if $g_{\alpha\beta} = g_{(\alpha\beta)}$, $F_{\mu\nu} = H_{\mu\nu}$). In this way $H_{\mu\nu}$ can be considered as an induction tensor of an electromagnetic field. Moreover, the second pair of Maxwell equations (1.46) suggests that rather $H^{\mu\nu}$ should be considered as an induction tensor. It is easy to see that if we take $g_{[\mu\nu]} = F_{\mu\nu} = 0$, $g_{(\alpha\beta)} = \eta_{\alpha\beta}$ (a Minkowski tensor) we satisfy field equations, i.e. Eqs (1.39)–(1.43). It means an empty Minkowski space is a solution of the equations.

It is easy to see that the theory contains GR as a limit $g_{[\mu\nu]} = 0$. In this case we get Einstein equations with electromagnetic sources. If we consider a Nonsymmetric Kaluza–Klein Theory with external sources (see Ref. [4]), we recover GR in the limit $g_{[\mu\nu]} = 0$. The theory satisfies a Bohr correspondence principle to GR. Thus we recover all the achievements of GR, e.g. Newton law, post Newtonian corrections, gravitational waves. Moreover, in our theory we have gravito-electromagnetic waves which are more general (Section 2). Post Newtonian approximation in the Nonsymmetric Kaluza–Klein Theory with material (external) sources can be done similarly as in NGT.

Our theory contains an antisymmetric field $g_{[\mu\nu]}$. Moreover it does not results in ghosts (see Ref. [8] and references cited therein, especially Ref. [9]). One of these five theories of gravity is

Moffat's NGT (see Ref. [7]) in real version. Our approach on the level of ghost consideration corresponds to NGT. It means after a linearization our theory does not differ from NGT and we can apply results from Ref. [9].

In the theory we get an electromagnetic field lagrangian

$$\mathcal{L}_{\text{em}} = -\frac{1}{8\pi}(H^{\mu\nu}F_{\mu\nu} - 2(g^{[\mu\nu]}F_{\mu\nu})^2) \quad (1.51)$$

which can be written in the form

$$\mathcal{L}_{\text{em}} = -\frac{1}{8\pi}((g^{\mu\alpha}g^{\nu\beta} - g^{\nu\beta}\tilde{g}^{(\mu\alpha)} + g^{\nu\beta}g^{\mu\omega}\tilde{g}^{(\tau\alpha)}g_{\omega\tau})F_{\alpha\beta}F_{\mu\nu} - 2(g^{[\mu\nu]}F_{\mu\nu})^2) \quad (1.52)$$

or

$$\mathcal{L}_{\text{em}} = -\frac{1}{8\pi}(F^{\mu\nu}F_{\mu\nu} - 2(g^{[\mu\nu]}F_{\mu\nu})^2 + (g^{\nu\beta}g^{\mu\omega}\tilde{g}^{(\tau\alpha)}g_{\omega\tau} - g^{\nu\beta}\tilde{g}^{(\mu\alpha)})F_{\alpha\beta}F_{\mu\nu}) \quad (1.53)$$

where

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}.$$

Let us consider energy-momentum tensor of an electromagnetic field in the Nonsymmetric Kaluza-Klein Theory, i.e. $T_{\alpha\beta}^{\text{em}}$. Using Eq. (1.48) one gets

$$T_{\alpha\beta}^{\text{em}} = \overset{\circ}{T}_{\alpha\beta} + \frac{1}{4\pi}t_{\alpha\beta} \quad (1.54)$$

where

$$\overset{\circ}{T}_{\alpha\beta} = \frac{1}{4\pi}\left(F^\tau{}_\alpha F_{\tau\beta} - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right) \quad (1.55)$$

is an energy-momentum tensor of an electromagnetic field in N.G.T.,

$$F^\tau{}_\alpha = g^{\tau\gamma}F_{\gamma\alpha} = -F_\alpha{}^\tau \quad (1.56)$$

and

$$t_{\alpha\beta} = g_{\gamma\beta}F_\nu{}^\tau F_{\omega\tau}g^{\varepsilon\gamma}\tilde{g}^{(\rho\nu)}\tilde{g}^{(\delta\omega)}g_{[\alpha\rho]}g_{[\varepsilon\delta]} - g_{\gamma\beta}\tilde{g}^{(\rho\nu)}(F^{\mu\gamma}F_{\nu\mu}g_{[\alpha\rho]} + F_{\mu\varepsilon}F_\nu{}^\mu g^{\varepsilon\gamma}g_{[\alpha\rho]}) - 2g^{[\mu\nu]}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{4}g_{\alpha\beta}\left(2(g^{[\mu\nu]}F_{\mu\nu})^2 - (g^{\nu\delta}g^{\mu\omega}\tilde{g}^{(\tau\varepsilon)}g_{\omega\tau} - g^{\nu\delta}\tilde{g}^{(\mu\varepsilon)})F_{\varepsilon\delta}F_{\mu\nu}\right) \quad (1.57)$$

is a correction coming from the Nonsymmetric Kaluza-Klein Theory.

Let us consider the second pair of Maxwell equations in the Nonsymmetric Kaluza-Klein Theory. One writes them in the following form (using (1.48)):

$$\bar{\nabla}_\mu F^{\alpha\mu} = J_p^\alpha + J^\alpha \quad (1.58)$$

where J^α is a topological current and J_p^α is a polarization current

$$\begin{aligned} J_p^\alpha &= 4\pi\bar{\nabla}_\mu M^{\alpha\mu} = \frac{4\pi}{\sqrt{-g}}\partial_\mu(\sqrt{-g}M^{\alpha\mu}) = \bar{\nabla}_\mu\left(g^{\alpha\beta}g^{\mu\gamma}\tilde{g}^{(\tau\rho)}(F_{\rho\gamma}g_{[\beta\tau]} - F_{\rho\beta}g_{[\gamma\tau]})\right) \\ &= \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\alpha\beta}g^{\mu\gamma}\tilde{g}^{(\tau\rho)}(F_{\rho\gamma}g_{[\beta\tau]} - F_{\rho\beta}g_{[\gamma\tau]})\right). \end{aligned} \quad (1.59)$$

The energy momentum tensor in the form (1.54) and the second pair of Maxwell equations can be obtained directly from Palatini variation principle with respect to $\overline{W}^\alpha_\beta$, $g_{\mu\nu}$ and A_μ for

$$R(W) = R(\overline{W}) + 8\pi\mathcal{L}_{\text{em}} \quad (1.60)$$

where \mathcal{L}_{em} is given by Eq. (1.53).

Writing as usual

$$F_{\mu\nu} = \begin{pmatrix} 0 & -B_3 & B_2 & -E_1 \\ B_3 & 0 & -B_1 & -E_2 \\ -B_2 & B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix} \quad (1.61)$$

$$H^{\mu\nu} = \begin{pmatrix} 0 & -H^3 & H^2 & -D^1 \\ H^3 & 0 & -H^1 & -D^2 \\ -H^2 & H^1 & 0 & -D^3 \\ D^1 & D^2 & D^3 & 0 \end{pmatrix} \quad (1.62)$$

and introducing Latin indices $a, b = 1, 2, 3$ we get

$$E_a = F_{4a}, \quad D^a = H^{4a} \quad (1.63)$$

$$\vec{D} = (D^1, D^2, D^3), \quad \vec{E} = (E_1, E_2, E_3) \quad (1.64)$$

$$\vec{B} = -(F_{23}, F_{31}, F_{12}), \quad \vec{H} = -(H^{23}, H^{31}, H^{12}) \quad (1.65)$$

or

$$B_a = -\frac{1}{2}\varepsilon_a{}^{bc}F_{bc}, \quad F_{cm} = -\varepsilon_{cm}{}^e B_e \quad (1.66)$$

$$H^a = -\frac{1}{2}\varepsilon^a{}_{bc}H^{bc}, \quad H^{cm} = -\varepsilon^{cm}{}_e H^e. \quad (1.67)$$

ε_{abc} is a usual 3-dimensional antisymmetric symbol. $\varepsilon_{123} = 1$ and it is unimportant for it if its indices are in up or down position. We keep those indices in up and down position only for a convenience.

Using Eq. (1.48) one gets

$$D^a = A^{ac}E_c + C^{ad}B_d \quad (1.68)$$

$$H^a = \overline{A}^{ac}E_c + \overline{C}^{ad}B_d. \quad (1.69)$$

In formulae (1.68) and (1.69) A^{ac} can be identified with ε_{ac} (a dielectric constant tensor) and \overline{C}^{ab} with $(\mu^{-1})_{ab}$ (an inverse of magnetic constant tensor). Remaining coefficients have more complex interpretation. Moreover, it is possible to think about them as on material properties of some kind generalized medium. A medium with nonzero C^{ad} and \overline{C}^{ad} is called *bianisotropic*. In our case they are induced by the nonsymmetric tensor $g_{\alpha\beta}$, where

$$A^{me} = g^{\mu e} g_{[\delta\mu]} \tilde{g}^{(4\delta)} g^{m4} - g^{44} (g_{[\delta\mu]} g^{\mu e} \tilde{g}^{(m\delta)} - g^{me}) + g^{\omega 4} g_{[\delta\omega]} g^{4e} \tilde{g}^{(m\delta)} - g^{m4} g^{4e} \quad (1.70)$$

$$C^{pe} = \varepsilon_{mz}{}^p (g^{z4} g^{me} + g^{\mu e} g_{[\delta\mu]} g^{z4} \tilde{g}^{(m\delta)} - g^{\omega 4} g_{[\delta\omega]} g^{me} \tilde{g}^{(z\delta)}) \quad (1.71)$$

$$\overline{A}^{pm} = \frac{1}{2}\varepsilon^p{}_{ek} (g^{4k} g^{me} - g^{mk} g^{4e} - (g^{mk} \tilde{g}^{(4\delta)} + g^{4k} \tilde{g}^{(m\delta)}) g^{\mu e} g_{[\delta\mu]} - g^{\omega k} g_{[\delta\omega]} g^{me} \tilde{g}^{(4\delta)}) \quad (1.72)$$

$$\overline{C}^{pf} = \frac{1}{2}\varepsilon^p{}_{ek} \varepsilon_{wm}{}^f (g^{wk} g^{me} - g^{wk} g^{\mu e} g_{[\delta\mu]} \tilde{g}^{(m\delta)} - g^{\omega k} g_{[\delta\omega]} g^{me} \tilde{g}^{(w\delta)}) \quad (1.73)$$

Let us come back to the formula (1.20) and transform it to the form

$$g_{\sigma\rho}g_{\delta\beta}g^{\gamma\delta}g^{\alpha\sigma}H_{\gamma\alpha} + H_{\beta\rho} = 2F_{\beta\rho}. \quad (1.74)$$

Eq. (1.74) gives us $F_{\beta\rho}$ —a strength of an electromagnetic field in term of $H_{\beta\rho}$ —an induction tensor. Moreover, we know that an induction tensor is rather given by Eq. (1.27). Thus we get

$$F_{\beta\rho} = \frac{1}{2}(g_{\mu\beta}g_{\nu\rho} + g_{\sigma\rho}g_{\delta\beta}g^{\gamma\delta}g^{\alpha\sigma}g_{\mu\gamma}g_{\nu\alpha})H^{\mu\nu}. \quad (1.75)$$

After some calculations one gets

$$B_a = K_{ae}H^e + L_{an}D^n \quad (1.76)$$

$$E_r = \overline{K}_{re}H^e + \overline{L}_{rn}D^n, \quad (1.77)$$

where

$$K_{ae} = \frac{1}{4}\varepsilon_a^{br}(g_{mb}g_{nr} + g_{\sigma r}g_{\delta b}g^{\gamma\delta}g^{\alpha\sigma}g_{m\gamma}g_{n\alpha})\varepsilon^{mn}_e \quad (1.78)$$

$$L_{an} = \frac{1}{4}\varepsilon^{br}_a[(g_{nb}g_{4r} + g_{\sigma r}g_{\delta b}g^{\gamma\delta}g^{\alpha\sigma}g_{n\gamma}g_{4\alpha}) - (g_{4b}g_{nr} + g_{\sigma r}g_{\delta b}g^{\gamma\delta}g^{\alpha\sigma}g_{4\gamma}g_{n\alpha})] \quad (1.79)$$

$$\overline{K}_{re} = \frac{1}{2}(g_{m4}g_{nr} + g_{\sigma r}g_{\delta 4}g^{\gamma\delta}g^{\alpha\sigma}g_{m\gamma}g_{n\alpha})\varepsilon^{mn}_e \quad (1.80)$$

$$\overline{L}_{rn} = \frac{1}{2}[(g_{44}g_{nr} + g_{\sigma r}g_{\delta 4}g^{\gamma\delta}g^{\alpha\sigma}g_{4\gamma}g_{n\alpha}) - (g_{n4}g_{4r} + g_{\sigma r}g_{\delta 4}g^{\gamma\delta}g^{\alpha\sigma}g_{n\gamma}g_{4\alpha})]. \quad (1.81)$$

In this way we get \vec{B} and \vec{E} in terms of \vec{H} and \vec{D} , similarly to the formulae (1.70)–(1.73). They are in some sense dual to those formulae.

Moreover, in Ref. [10] one can find an induction between electric and magnetic fields caused by topological effects. Another connection between our work and Ref. [10] is using a dimensional reduction which is close to the Kaluza–Klein idea.

In our theory the nonsymmetric tensor $g_{\mu\nu}$ (a nonsymmetric gravitational field) results as a bianisotropic linear medium. Due to this we get Eqs (1.68)–(1.69). Moreover, in classical electromagnetic theory there are some formalisms for linear bianisotropic phenomena (see Refs [11], [12], [13], [14]). The authors consider classical electrodynamics of continuous media in several forms including a covariant form, i.e.

$$H^{\mu\nu} = \kappa^{\mu\nu\alpha\beta}F_{\alpha\beta} \\ \text{or } F_{\alpha\beta} = \overline{\kappa}_{\alpha\beta\mu\nu}H^{\mu\nu}$$

(see Ref. [11]), similar to our Eqs (1.48) and (1.20).

In the theory of spatially dispersive materials one can find the following formulae written in dyadic formalism,

$$\vec{D} = \overline{\varepsilon} \cdot \vec{E} + \overline{\xi} \cdot \vec{H} \quad \text{and} \quad \vec{B} = \overline{\mu} \cdot \vec{H} - \overline{\xi}^T \cdot \vec{E},$$

where $\overline{\varepsilon}$ is the permittivity and $\overline{\mu}$ is the permeability depending on space (and time). $\overline{\xi}$ is one more parameter for mirror-asymmetric structure (chiral materials). For magnetoelectric coupling media one gets

$$\vec{D} = \overline{\varepsilon} \cdot \vec{E} + \overline{\xi} \cdot \vec{H} \quad \text{and} \quad \vec{B} = \overline{\mu} \cdot \vec{H} + \overline{\xi}^T \cdot \vec{E},$$

where $\overline{\zeta}$ is one more parameter which has to do with parity property of the material with respect to time inversion. These equations are similar to our Eqs (1.68)–(1.69) (see Ref. [12]). In Refs [13], [14] the authors are using differential forms in classical electrodynamics also for bianisotropic media (see Ref. [14], also [15]), getting similar formulae in covariant form. In Ref. [13] the authors consider also nonlinear electrodynamics by Born–Infeld (see Ref. [16]) and also in a more general Plebański’s form (see Ref. [17]). In our approach constitutive relations are linear (in nonlinear electrodynamics they are nonlinear), but field equations are nonlinear. In Ref. [12] one considers several currents, e.g. external, polarization. In our case we have also several currents, i.e. a polarization current and topological current, see Eq. (1.58) and (1.59) and (1.47). The covariant form of the linear constitutive equations has been considered in a general form in Ref. [11].

In GR we have also a tensor $H^{\mu\nu}$ given by the formula

$$H^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$$

where $g^{\mu\alpha}$ is an inverse tensor of a symmetric metric in GR. In this way we can have an induction tensor $H^{\mu\nu}$ in curvilinear coordinates in space (e.g. spherical). Someone can define an induction “tensor” in a different way, as a tensor density, i.e.

$$\tilde{h}^{\mu\nu} = \sqrt{-g} H^{\mu\nu}.$$

We do not follow this approach. In this way we have to do with bianisotropic medium in GR and also in flat Minkowski space in curvilinear coordinates, i.e.

$$\begin{aligned} D^a &= (g^{44} g^{ab} - g^{4b} g^{a4}) E_a - (g^{4m} g^{an} - g^{4n} g^{am}) \varepsilon_{mn}{}^e B_e \\ H^a &= \frac{1}{2} \varepsilon^a{}_{mn} (g^{mb} g^{n4} - g^{m4} g^{nb}) E_b + \frac{1}{2} \varepsilon^a{}_{mn} \varepsilon_{cb}{}^e g^{[m[c} g^{n]b]}. \end{aligned}$$

It is easy to see that the “medium” is bianisotropic if $g^{4m} \neq 0$.

In the Nonsymmetric Kaluza–Klein Theory the situation is more complex:

$$H^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} H_{\alpha\beta}$$

and $H_{\alpha\beta}$ is given by the formula (1.48).

One can also consider a tensorial density

$$\tilde{h}^{\mu\nu} = \sqrt{-g} H^{\mu\nu}.$$

Moreover, we do not follow this approach. In Refs [18], [19] one can find some conditions posed on ε_{ac} (or $\varepsilon\delta_{ac}$) and $(\mu^{-1})_{ab}$ (or $\frac{\delta_{ab}}{\mu}$). In the Nonsymmetric Kaluza–Klein Theory such conditions are not satisfied for our “generalized medium” is a gravitational field described by the nonsymmetric tensor $g_{\mu\nu}$. Let us notice the following fact: even if $g^{4m} = 0 \neq g^{m4}$ (in the nonsymmetric case) our constitutive relations can still describe bianisotropic medium.

Let us notice that in the case of a diagonal $g_{(\alpha\beta)}$, $F_{\mu\nu} = H_{\mu\nu}$, also in the case of spherically symmetric $g_{\mu\nu}$ we have $F_{\mu\nu} = H_{\mu\nu}$, which was extensively used in order to find an exact solution for field equations (see [4]).

Formulae (1.61)–(1.62) have a formal character. The physical meaning of (\vec{D}, \vec{H}) and (\vec{E}, \vec{B}) as induction or strength vectors of electric or magnetic fields is sound only in a stationary case.

Let us consider a nonsymmetric tensor for axially symmetric and stationary space-time in cylindrical coordinates (see Ref. [20])

$$g_{\mu\nu} = \begin{pmatrix} -e^{2(n-l)} & 0 & ade^n & de^n \\ 0 & -e^{2(n-l)} & kae^m & ke^n \\ -ade^n & -ake^n & ca^2e^{2l} - r^2e^{-2l} & ace^{2l} \\ -de^n & -ke^n & ace^{2l} & ce^{2l} \end{pmatrix} \quad (1.82)$$

where $c = 1 + d^2 + k^2$, $x^1 = r$, $x^2 = z$, $x^3 = \theta$, $x^4 = t$, and all the functions n, l, a, b, d, k are functions of r and z only,

$$g = r^2e^{4(n-l)}, \quad \tilde{g} = -r^2e^{4(n-l)}(1 + d^2 + k^2). \quad (1.83)$$

An electromagnetic field is described by

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & p & s \\ 0 & 0 & q & u \\ -p & -q & 0 & 0 \\ -s & -u & 0 & 0 \end{pmatrix} \quad (1.84)$$

$$p = B_z, \quad s = -E_r, \quad q = -B_r, \quad u = -E_z,$$

and all the functions depend on r and z only.

In this case we can calculate $H_{\mu\nu}$ and $H^{\mu\nu}$. One gets

$$H_{\mu\nu} = \frac{1}{\tilde{g}} \begin{pmatrix} 0 & r^2m_1 & -pr^2m_2 & -r^2sm_2 \\ -r^2m_1 & 0 & -qr^2m_2 & -r^2um_2 \\ pr^2m_2 & qr^2m_2 & 0 & r^2m_3 \\ r^2sm_2 & r^2um_2 & -r^2m_3 & 0 \end{pmatrix} \quad (1.85)$$

where $m_1 = (du - ks)e^{5n-6l}$, $m_2 = (1 + d^2 + k^2)e^{4(n-l)}$, $m_3 = (d(p - as) + k(q - as))e^{3n-2l}$,

$$H^{\mu\nu} = \frac{1}{P} \begin{pmatrix} 0 & 0 & r^4m_4 & ar^4m_4 + r^6m_6 \\ 0 & 0 & r^4m_5 & -ar^4m_5 + r^6m_7 \\ -r^4m_4 & -r^4m_5 & 0 & 0 \\ -ar^4m_4 - r^6m_6 & ar^4m_5 - r^6m_7 & 0 & 0 \end{pmatrix} \quad (1.86)$$

where

$$\begin{aligned} P &= e^{12(n-l)}r^6(1 + d^2 + k^2), \\ m_4 &= e^{10n-8l}(1 + d^2 + k^2)(as - p), \quad m_5 = e^{10n-8l}(1 + d^2 + k^2)(au - q), \\ m_6 &= e^{10n-12l}(s + d^2s + dku), \quad m_7 = e^{10n-12l}(dks + u + k^2u). \end{aligned} \quad (1.87)$$

Let us come back to Eq. (1.32) (the second part of Maxwell equations) and let us consider it for $\alpha = 4$

$$\partial_m \underline{H}^{4m} = 2g^{[4b]} \partial_b (g^{[\mu\nu]} F_{\mu\nu}), \quad m, b = 1, 2, 3, \quad (1.88)$$

or

$$\text{div}(\sqrt{-g} \vec{D}) = \rho \sqrt{-g}. \quad (1.89)$$

In this equation ρ represents a density of an electric charge.

If $\vec{D} = 0$ the density of charge equals zero. The problem which we now pose is as follows. Is it possible to have $\vec{D} = 0$ and $\vec{E} \neq 0$? This means that we have a condition

$$A^{ae} E_e + C^{ae} H_e = 0. \quad (1.90)$$

This means we have a confinement of a charge induced by a special properties of “vacuum” (i.e. a gravitational field described by nonsymmetric tensor $g_{\mu\nu}$). It means non-zero electric field and zero charge distribution.

In the case of a stationary, axially symmetric field we have conditions

$$as - p = \frac{r^2(s + d^2s + dku)}{a(1 + d^2 + k^2)} e^{-4l} \quad (1.91)$$

$$au - q = \frac{r^2(dks + u + k^2u)}{a(1 + d^2 + k^2)} e^{-4l} \quad (1.92)$$

with $u, s \neq 0$.

These conditions can be imposed on functions n, l, d, k, a to be satisfied for any nonzero u, s with some dependence on p and q . One gets

$$\begin{aligned} e^{4l} a^2 (1 + d^2 + k^2) &= r^2 (1 + d^2) \\ p &= -u \frac{r^2 dk}{a(1 + d^2 + k^2)} e^{-4l} \\ e^{4l} a^2 (1 + d^2 + k^2) &= r^2 (1 + k^2) \\ q &= -s \frac{dkr^2}{a(1 + d^2 + k^2)} e^{-4l} \end{aligned} \quad (1.93)$$

and finally

$$d = k, \quad a = r e^{-2l} \sqrt{\frac{1 + d^2}{1 + 2d^2}}, \quad (1.94)$$

$$p = -\beta u, \quad q = -\beta s, \quad (1.95)$$

where

$$\beta = \frac{r d e^{2l}}{\sqrt{(1 + d^2)(1 + 2d^2)}} \quad (1.96)$$

or

$$B_z = \beta E_z \quad (1.97)$$

$$B_r = -\beta E_r. \quad (1.98)$$

We do not give here any quantum version of the theory. It is a classical field theory and a classical theory of a charge confinement. Moreover, a confinement is a nonperturbative effect and cannot be obtained in perturbative quantum field theory. This dielectric model of a charge confinement can be considered as an “interference effect” between gravity and electromagnetism

in our unified classical field theory. In order to find a quantum version of the theory we should consider Ashtekar–Lewandowski canonical quantization procedure of the theory, which is suitable here for the theory is nonlinear and contains gravity (see Ref. [21]).

According to the Einstein programme of geometrization and unification of physical interactions we should get equations where on the left-hand side we have geometrical quantities and on the right-hand side material quantities. A full programme is completed if on the right-hand side we get zero. It means all quantities have been geometrized. Eqs (1.29)–(1.32) give us in this sense geometrization and unification of gravity and electromagnetism if we shift $8\pi T_{\alpha\beta}^{\text{em}}$ from the right-hand side to the left in Eq. (1.29) and $2g^{[\alpha\beta]}\partial_\beta(g^{[\mu\nu]}F_{\mu\nu})$ from the right-hand side to the left in Eq. (1.32). Having in mind Eq. (1.33) we see that $8\pi T_{\alpha\beta}^{\text{em}}$ is geometrized. According to this geometrization and unification programme all quantities coming from higher dimension should get an interpretation in terms of matter defined on the space-time. In this way in the theory we geometrized all quantities completing Einstein programme for unification of gravity and electromagnetism getting “interference effects” between both interactions.

2 Gravito–electromagnetic waves solutions in the Nonsymmetric Kaluza–Klein Theory

Let us consider the following nonsymmetric metric in cartesian coordinates

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & r & -r \\ 0 & -1 & s & -s \\ -r & -s & e-1 & -e \\ r & s & -e & 1+e \end{pmatrix} \quad (2.1)$$

where

$$e = e(x, y, z - t)$$

$$s = s(x, y, z - t)$$

$$r = r(x, y, z - t)$$

which describes generalized plane wave for $g_{\mu\nu}$ and

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & k & -k \\ 0 & 0 & p & -p \\ -k & -p & 0 & 0 \\ k & p & 0 & 0 \end{pmatrix} \quad (2.2)$$

where

$$p = p(x, y, z - t)$$

$$q = q(x, y, z - t)$$

which describes generalized plane electromagnetic wave. All the functions mentioned here are subject of field equations in Nonsymmetric Kaluza–Klein Theory. Using results from Ref. [22] one gets the following equations

$$-\Delta e + 4Q - \left[\left(\frac{\partial r}{\partial x} - \frac{\partial s}{\partial y} \right)^2 + \left(\frac{\partial r}{\partial y} + \frac{\partial s}{\partial x} \right)^2 \right] = 4 \left[\left(\frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial A}{\partial y} \right)^2 \right] \quad (2.3)$$

where

$$Q = \left[\left(\frac{\partial r}{\partial x} \right)^2 + r \frac{\partial^2 r}{\partial x^2} + \frac{1}{2} \frac{\partial s}{\partial x} \left(\frac{\partial r}{\partial y} + \frac{\partial s}{\partial x} \right) + \frac{1}{2} s \left(\frac{\partial^2 r}{\partial x \partial y} + \frac{\partial^2 s}{\partial x^2} \right) + \left(\frac{\partial s}{\partial y} \right)^2 + s \frac{\partial^2 s}{\partial y^2} + \frac{1}{2} \frac{\partial r}{\partial y} \left(\frac{\partial r}{\partial y} + \frac{\partial s}{\partial x} \right) + \frac{1}{2} r \left(\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 s}{\partial x \partial y} \right) \right] \quad (2.4)$$

$$\Delta A(x, y, z - t) = 0 \quad (2.5)$$

$$p = \frac{\partial A}{\partial x}, \quad q = -\frac{\partial A}{\partial y} \quad (2.6)$$

$$\frac{\partial r}{\partial y} = \frac{\partial s}{\partial x} + H(x, y, z - t), \quad \Delta H(x, y, z - t) = 0 \quad (2.7)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator in two dimensions. Equation (2.3) is a Poisson equation for e . Using Eq. (1.30) one gets

$$\frac{\partial s}{\partial y} = -\frac{\partial r}{\partial x} \quad \text{or} \quad s = -\frac{\partial B}{\partial x}, \quad r = \frac{\partial B}{\partial y}.$$

In this way we get

$$\Delta B = H. \quad (2.8)$$

A and H are arbitrary harmonic functions in two dimensions with arbitrary dependence on $(z - t)$ of C^2 class. B is an arbitrary solution of Poisson equation and arbitrary function for $(z - t)$ of C^2 class. The function e can be obtained from the Poisson equation

$$\Delta e = f \quad (2.9)$$

where f is given in terms of B, A, H and takes a simple form

$$f = -4 \left[\left(\frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial A}{\partial y} \right)^2 \right] + \left(\frac{\partial^2 B}{\partial y^2} - \frac{\partial^2 B}{\partial x^2} \right)^2 + 2 \left(\frac{\partial B}{\partial x} \frac{\partial H}{\partial x} + \frac{\partial B}{\partial y} \frac{\partial H}{\partial y} \right).$$

The dependence on $(z - t)$ is parametric and is given by the dependence on $(z - t)$ of B, A, H . This solution describes a generalized plane gravito-electromagnetic wave.

Now we consider the following nonsymmetric tensor in cartesian coordinates

$$g_{\mu\nu} = \begin{pmatrix} -a & 0 & r & -r \\ 0 & -a & s & -s \\ -r & -s & -b & 0 \\ r & s & 0 & b \end{pmatrix}, \quad (2.10)$$

$$a = a(x, y), \quad r = r(x, y, z - t),$$

$$b = b(x, y), \quad s = s(x, y, z - t),$$

and an electromagnetic field strength tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & p & -p \\ 0 & 0 & q & -q \\ -p & -q & 0 & 0 \\ p & q & 0 & 0 \end{pmatrix}, \quad (2.11)$$

$$p = p(x, y, z - t), \quad q = q(x, y, z - t). \quad (2.12)$$

We put (2.10) and (2.11) into the field equation (1.29)–(1.35). Using the results from Ref. [23] we get the following equations:

$$\Delta(\alpha/a) = 4a\beta^2 - \frac{2}{a} \left[\left(\frac{\partial\psi}{\partial x} \right)^2 + \left(\frac{\partial\psi}{\partial y} \right)^2 \right] \quad (2.13)$$

$$p = \frac{\partial\psi}{\partial y}, \quad q = -\frac{\partial\psi}{\partial x} \quad (2.14)$$

$$\psi = \psi(x, y, z - t), \quad \Delta\psi = 0$$

$$\beta = \frac{1}{2a} \left(\frac{\partial r}{\partial y} - \frac{\partial s}{\partial x} \right) \quad (2.15)$$

$$\alpha = r^2 + s^2. \quad (2.16)$$

From the remaining field equation we get

$$\Delta\beta = 0, \quad (2.17)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$,

$$a = e^A, \quad (2.18)$$

where A is an arbitrary harmonic function in two variables,

$$\Delta A = 0. \quad (2.19)$$

The function b is given by

$$b = G_1(z + t)G_2(z - t) \quad (2.20)$$

where G_1 and G_2 are arbitrary functions of C^2 class (of one variable). The function β should be written in the following form

$$\beta(x, y, z, t) = f_0(z - t)\beta_0(x, y) \quad (2.21)$$

where β_0 is an arbitrary harmonic function and f_0 is an arbitrary function of one variable (of C^2 class). Let us consider Eq. (1.30). One gets

$$\frac{\partial r}{\partial y} + \frac{\partial s}{\partial x} = 0. \quad (2.22)$$

From this equation we get

$$r = \frac{\partial\varphi}{\partial x}, \quad s = -\frac{\partial\varphi}{\partial y} \quad (2.23)$$

where φ is a function (arbitrary) of two variables (of C^3) and a function of $z - t$.

One gets

$$\Delta\varphi = 2e^A f_0 \beta_0. \quad (2.24)$$

Let $\tilde{\varphi}$ be any arbitrary solution of Eq. (2.24) (remembering that A , f_0 and β_0 are arbitrary). In this way

$$\alpha = r^2 + s^2 = \left(\frac{\partial\tilde{\varphi}}{\partial y} \right)^2 + \left(\frac{\partial\tilde{\varphi}}{\partial x} \right)^2. \quad (2.25)$$

Let us consider Eq. (2.13) in the following form

$$\Delta g = 4e^A \beta_0^2 f_0^2 - \frac{2}{e^A} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] \quad (2.26)$$

and let \tilde{g} be any solution of this Poisson equation.

Thus

$$\frac{\alpha}{a} = \tilde{g} \quad (2.27)$$

$$\left(\frac{\partial \tilde{\varphi}}{\partial y} \right)^2 + \left(\frac{\partial \tilde{\varphi}}{\partial x} \right)^2 = \tilde{g} e^A. \quad (2.28)$$

Eq. (2.28) gives us a consistency condition for an existence of gravito-electromagnetic wave.

Let us consider the following nonsymmetric metric

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & b & 0 & l+q \\ 0 & 0 & b & m+p \\ 1 & l-q & m-p & -v \end{pmatrix} \quad (2.29)$$

and an electromagnetic field strength tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & u \\ 0 & -s & -u & 0 \end{pmatrix} \quad (2.30)$$

in cartesian coordinates

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = t.$$

It is possible to consider x^4 as $z - t$. We suppose that

$$\frac{\partial}{\partial x^1} g_{\mu\nu} = \frac{\partial}{\partial x^1} F_{\mu\nu} = 0.$$

Using results from Refs [24], [25] and field equation of the Nonsymmetric Kaluza–Klein Theory we get the following equations

$$\begin{aligned} s &= \frac{\partial \psi}{\partial x^3}, \\ u &= \frac{\partial \psi}{\partial x^2}, \\ \left(\frac{\partial^2}{\partial (x^2)^2} + \frac{\partial^2}{\partial (x^3)^2} \right) \psi &= 0, \quad \psi = \psi(x^2, x^3, x^4). \end{aligned} \quad (2.31)$$

Supposing

$$b = 1, \quad q = 0 \quad (2.32)$$

we get further

$$\begin{aligned}
p &= p(x^2, x^4) \\
\frac{\partial^3}{\partial(x^2)^3} p &= 0 \\
\frac{\partial l}{\partial x^3} &= \frac{\partial m}{\partial x^2}, \quad \frac{\partial l}{\partial x^2} = -\frac{\partial m}{\partial x^3} \\
l &= l(x^2, x^3, x^4), \quad m = m(x^2, x^3, x^4).
\end{aligned} \tag{2.33}$$

Thus l and m are harmonically conjugate.

Writing

$$v = 1 + f, \quad f = f(x^2, x^3, x^4), \tag{2.34}$$

where f satisfies the equation

$$\Delta f = 2p^2 \left(\frac{p_{,22}}{p} + \left(\frac{p_{,2}}{p} \right)^2 \right) - 2((\psi_{,2})^2 + (\psi_{,3})^2), \tag{2.35}$$

the form of p can be easily found

$$p = \frac{(x^2)^2}{2} \varphi_1(x^4) + x^2 \varphi_2(x^4) + \varphi_3(x^4)$$

where φ_i , $s = 1, 2, 3$, are arbitrary functions of one variable of C^3 class.

This solution represents a gravito-electromagnetic wave if we interpret x^4 as wave front variable $z - t$. “,” means a derivative with respect to x^2 or x^3 , f is a solution of a Poisson equation with a parametric dependence on x^4 imposed by ψ , φ_i , $i = 1, 2, 3$. Functions ψ , l and m are arbitrary functions of x^4 variable of C^2 class.

For all three solutions considered here $t_{\mu\nu} = M_{\mu\nu} = J^\mu = J^\mu_p = 0$. Moreover, $\vec{D} \neq \vec{E}$ for all solutions. Those solutions are not solutions of nonsymmetric theory of gravity coupled to a Maxwell field. They are examples of wave solutions in the Nonsymmetric Kaluza–Klein Theory, a unified theory of gravity and an electromagnetic field.

3 An influence of a cosmological constant on a solution in the Nonsymmetric Kaluza–Klein Theory

Let us consider Eqs. (1.29)–(1.35) and let us change $\overset{\text{em}}{T}_{\alpha\beta}$ into

$$T_{\alpha\beta} = \overset{\text{em}}{T}_{\alpha\beta} - \frac{\Lambda}{8\pi} g_{\alpha\beta}.$$

It means we introduce a cosmological constant Λ to the theory.

In the Nonsymmetric Kaluza–Klein Theory $\Lambda = 0$. Moreover, in the Nonsymmetric Non-abelian Kaluza–Klein Theory this constant is in general non-zero. The Yang–Mills field can be reduced to a $U(1)$ subgroup of a group G (see Ref. [3] for more details). We also consider Nonsymmetric Kaluza–Klein Theory with external sources, and a cosmological constant term can be added to the external energy momentum tensor (see Ref. [4]).

Let us consider corrected field equations in a static, spherically symmetric case. Using results from Ref. [4] we get the following exact solution

$$g_{\mu\nu} = \begin{pmatrix} -\alpha & 0 & 0 & \omega \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2 \theta & 0 \\ -\omega & 0 & 0 & \gamma \end{pmatrix} \quad (3.1)$$

where

$$\omega = \frac{l^2}{r^2} \quad (3.2)$$

$$\alpha^{-1} = \left(1 + \frac{Q^2}{\bar{b}r} g\left(\frac{r}{\bar{b}}\right) + \frac{\Lambda r^2}{3} \right) \quad (3.3)$$

$$\gamma = \left(1 + \frac{l^4}{r^4} \right) \left(1 + \frac{Q^2}{\bar{b}r} g\left(\frac{r}{\bar{b}}\right) + \frac{\Lambda r^2}{3} \right) \quad (3.4)$$

$$F_{14} = E = -\frac{Q}{r^2} \left(\frac{r^4}{r^4 + \bar{b}^4} \right) \quad (3.5)$$

$$\bar{b}^4 = 4l^4 \quad (3.6)$$

Q is an electric charge and l is an integration constant of length dimension. Let us notice that a solution with a cosmological constant differs from the previous solution (see Ref. [4]) only by a term $\frac{\Lambda r^2}{3}$ similarly as in General Relativity for a Schwarzschild solution with a cosmological constant.

Similar solution is known in Einstein Unified Field Theory and in Schrödinger Theory (see Ref. [26]) (i.e. which generalizes Schwarzschild-like solution from those theories to the case with non-zero cosmological constant). This means that cosmological constant results only via term $\frac{\Lambda r^2}{3}$. Thus the solution is not asymptotically flat.

It is easy to see that

$$F_{14} = E \xrightarrow{r \rightarrow \infty} -\frac{Q}{r^2} \quad (3.7)$$

from Eq. (3.5). From Eq. (3.4) and the definition of the function $g(x)$ (see Eq. (3.13) below) one gets

$$\alpha^{-1} \xrightarrow{r \rightarrow \infty} 1 - \frac{2m_N}{r} + \frac{Q^2}{r^2} + \frac{\Lambda r^2}{3} \quad (3.8)$$

where

$$m_N = \frac{\pi Q^2}{2\sqrt{2b}} \quad (3.9)$$

(see Ref. [4]) is an energy of a solution (a Schwarzschild-like or Kottler-like asymptotically). It is also easy to see that $\alpha^{-1}(0) = 1$, $F_{14}(0) = E(0) = 0$.

All the details concerning the solution without a cosmological constant can be found in Ref. [4]. In Ref. [4] it has been proved that the energy of the solution is finite and the total charge is equal to Q . An electric field of the solution is plotted in Fig. 3 of Ref. [4] (it is the same as in the case with non-zero cosmological constant).

Moreover, we repeat some details important for a reader. This solution is not a solution with a cosmological constant in a nonsymmetric theory of gravity coupled to a Maxwell field. This solution cannot be obtained in pure NGT with electromagnetic sources. Thus its remarkable properties concerning nonsingularity of electric and gravitational fields are “interference effects” between gravity and electromagnetism in our unification of gravity and electromagnetism. The solution asymptotically behaves as Reissner–Nordström solution in NGT. Due to this it satisfies Bohr correspondence principle between our unification (Nonsymmetric Kaluza–Klein Theory) and NGT and General Relativity. The solution achieves an old dream by Einstein, Weyl, Kaluza, Schrödinger, Eddington on a *unitary classical field theory* which has spherically-symmetric singularity-free solutions of the field equations treated as particles.

The properties of the solution are traced back to Abraham–Lorentz (see Refs [27], [28], [29], [30]) idea and more advanced in Born–Infeld electrodynamics (see Refs [16], [31]) of an electron being a particle-like finite-energy field configuration, the soliton, which energy is of completely field nature. It means it is an energy of field selfinteraction. The solution is in rest. In order to get a moving soliton it is enough to boost it via a Lorentz transformation.

Our theory is *nonlinear* (nonlinear field equations). Moreover, our constitutive equations between a field strength tensor $F_{\mu\nu}$ and an induction tensor $H^{\mu\nu}$ are *linear*.

Let us introduce the following notation

$$a = \frac{Q^2}{2l^2} \quad \left(a = \frac{G_N Q^2}{2c^2 l^2} \right) \quad (3.10)$$

$$b = \frac{2l^2 \Lambda}{3} \quad (3.11)$$

In this notation one gets

$$\alpha^{-1} = 1 + a \frac{g(x)}{x} + bx^2 = f(x) \quad (3.12)$$

where

$$g(x) = \frac{1}{4\sqrt{2}} \log \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) - \frac{1}{2\sqrt{2}} \left[\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1) \right] \quad (3.13)$$

(see Ref. [4]),

$$x = \frac{r}{\sqrt{2}l}. \quad (3.14)$$

The function $g(x)$ has been plotted in Fig. 2 of Ref. [4]. Similarly as in Ref. [4] one writes $f(x) = 1 - P(x)$. In this way $P(x)$ has an interpretation of a generalized Newtonian gravitational field in normalized radial coordinate, i.e. $P(x) = -a \frac{g(x)}{x} - bx^2$ (see Schwarzschild solution in GR, see also Fig. 6 of Ref. [4] in the case of $b = 0$). The solution described here can be considered as a classical model of an electron. We can try to quantize it using collective coordinate approach. One can also apply loop quantum gravity approach (see Ref. [21]) similarly as in Ref. [32].

Now it is interesting to examine an influence of the cosmological constant Λ (via b) on properties of the solution. The most interesting case is to find horizons for such a solution. It means, to find zeros of the function

$$f(x) = \alpha^{-1} = 1 + a \frac{g(x)}{x} + bx^2 = 0. \quad (3.15)$$

(The function $f(x)$ gives us information of gravitation field of the discussed solution. In the case of zero cosmological constant we have a plot in Fig. 6 of Ref. [4].)

All the plots of the function $f(x)$ for some values of parameters a and b give us some information on the behaviour of relativistic gravitational field of the solution for some critical values of a and b and some typical behaviour of the solution among critical values. For we expect horizons $f(x)$ gives us more than $P(x)$. We have of course $P(0) = 0$ ($f(0) = 1$).

In the previous case $b = 0$ (see Ref. [4]) we get that there is a critical value of $a = a_{\text{crt}} = 3.17\dots$ such that for $a < a_{\text{crt}}$ there is not any horizon, for $a = a_{\text{crt}}$ there is one horizon, and for $a > a_{\text{crt}}$ there are two horizons. Let us suppose that $\Lambda > 0$ ($b > 0$) and examine a_{crt} in this case. One gets

$$\begin{aligned} \text{for } b = 0.001, & \quad a_{\text{crt}} \cong 3.2 \\ \text{for } b = 0.01, & \quad a_{\text{crt}} \cong 3.6 \\ \text{for } b = 0.1, & \quad a_{\text{crt}} \cong 4 \\ \text{for } b = 1, & \quad a_{\text{crt}} \cong 11. \end{aligned}$$

It means that as before we have for $a < a_{\text{crt}}$ not any horizon, $a = a_{\text{crt}}$ one horizon and for $a > a_{\text{crt}}$ two horizons. Thus the cosmological constant results in a higher value of a_{crt} .

It is interesting to consider a negative value of Λ ($b < 0$). In this case we have always one more horizon (the so-called de Sitter horizon, a cosmological horizon). For example for $a = 0.1$, $b = -0.001$ we have only one horizon. For $a = 5$, $b = -0.001$ we have three horizons, two as before for $b > 0$ and one de Sitter horizon. For $a = 3$, $b = -0.01$ we have two horizons, one de Sitter horizon and one double as before for $b > 0$. In this case $a = 3$ is a critical value for $b = -0.001$. For lower negative values of b , i.e. $b = -0.1$, we have for $a = 0.1, 0.7, 3$ only one (de Sitter) horizon.

Summing up, for $\Lambda > 0$ we get two horizons, one horizon or no horizon, $0 < r_{H_1} < r_{H_2}$, $0 < r_H$ (as in the case of Reissner–Nordström solution). For $\Lambda < 0$ we have a de Sitter horizon

$$\begin{aligned} 0 &< r_S \\ 0 &< r_{H_1} < r_{H_2} < r_S \\ 0 &< r_H < r_S \end{aligned}$$

and inside it one or two horizons or a case without an inside horizon. In Figures 1, 2, 3, 4, 5 we give plots of the function $f(x)$ for several values of parameters a and b .

One gets a value of critical parameter a_{crt} and a critical value of a horizon radius from the following equations (let us remind to the reader that x is r (a radius) in a convenient unit $\sqrt{2}l$, see Eq. (3.14))

$$\begin{aligned} 0 &= f(x_{\text{crt}}) = 1 + a_{\text{crt}} \frac{g(x_{\text{crt}})}{x_{\text{crt}}} + bx_{\text{crt}}^2 \\ 0 &= \frac{df}{dx}(x_{\text{crt}}) = 2bx_{\text{crt}} + \frac{a_{\text{crt}}}{x_{\text{crt}}} \frac{dg}{dx}(x_{\text{crt}}) - \frac{a_{\text{crt}}}{x_{\text{crt}}^2} g(x_{\text{crt}}). \end{aligned} \tag{3.16}$$

After some algebra we get

$$x_{\text{crt}}^3(1 + bx_{\text{crt}}^2) + g(x_{\text{crt}})(x_{\text{crt}}^4 + 1)(3bx_{\text{crt}}^2 + 1) = 0 \tag{3.17}$$

$$a_{\text{crt}} = -\frac{(1 + bx_{\text{crt}}^2)x_{\text{crt}}}{g(x_{\text{crt}})} = \frac{(1 + x_{\text{crt}}^4)}{x_{\text{crt}}^2} (3bx_{\text{crt}}^2 + 1) \tag{3.18}$$

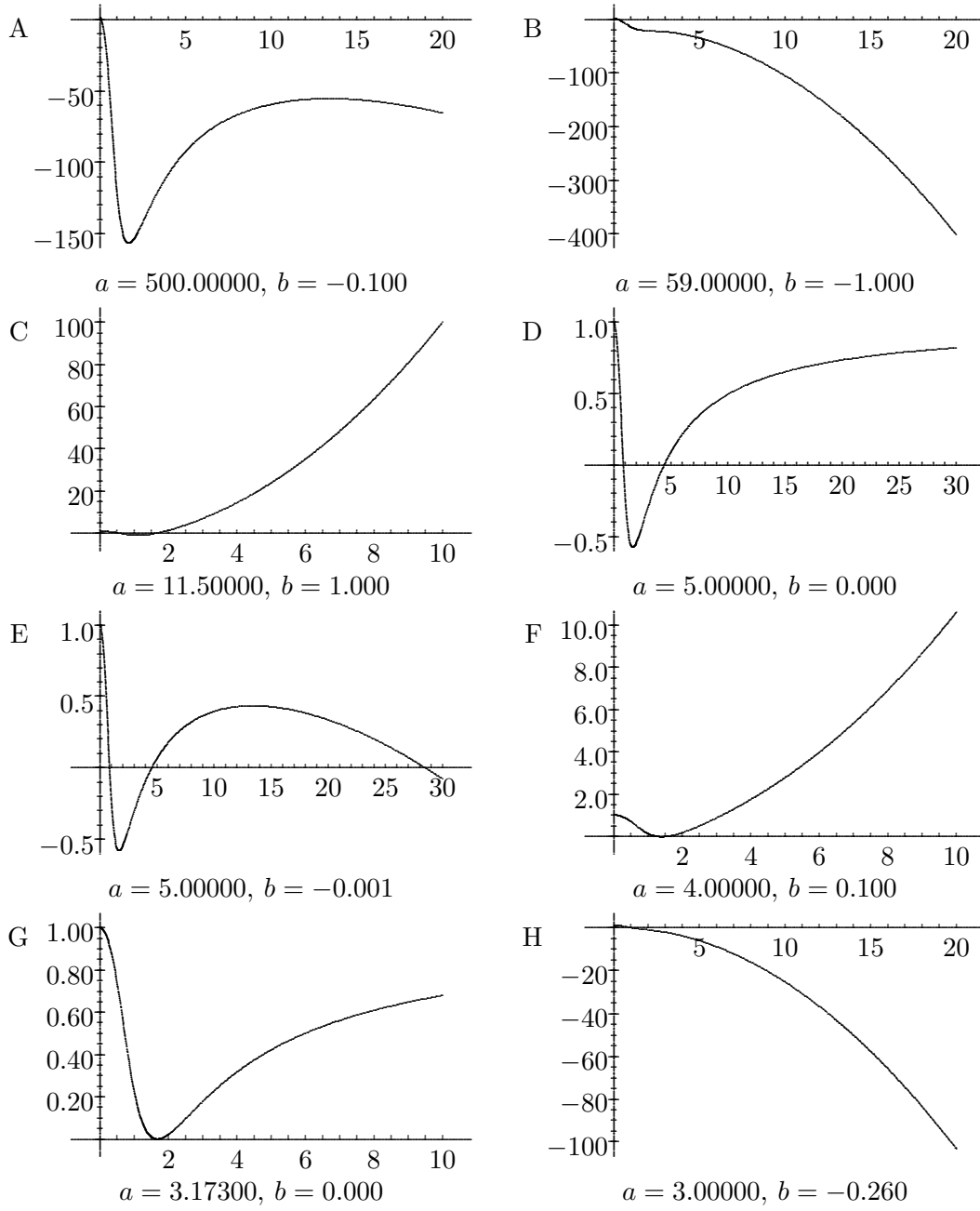


Fig. 1. Plots of the function $f(x)$ (Eq. (3.15)) for some values of parameters a and b (see the text for explanation)

Fig. 1A: $a = 500.00000$, $b = -0.100$; Fig. 1B: $a = 59.00000$, $b = -1.000$;
 Fig. 1C: $a = 11.50000$, $b = 1.000$; Fig. 1D: $a = 5.00000$, $b = 0.000$;
 Fig. 1E: $a = 5.00000$, $b = -0.001$; Fig. 1F: $a = 4.00000$, $b = 0.100$;
 Fig. 1G: $a = 3.17300$, $b = 0.000$; Fig. 1H: $a = 3.00000$, $b = -0.260$.

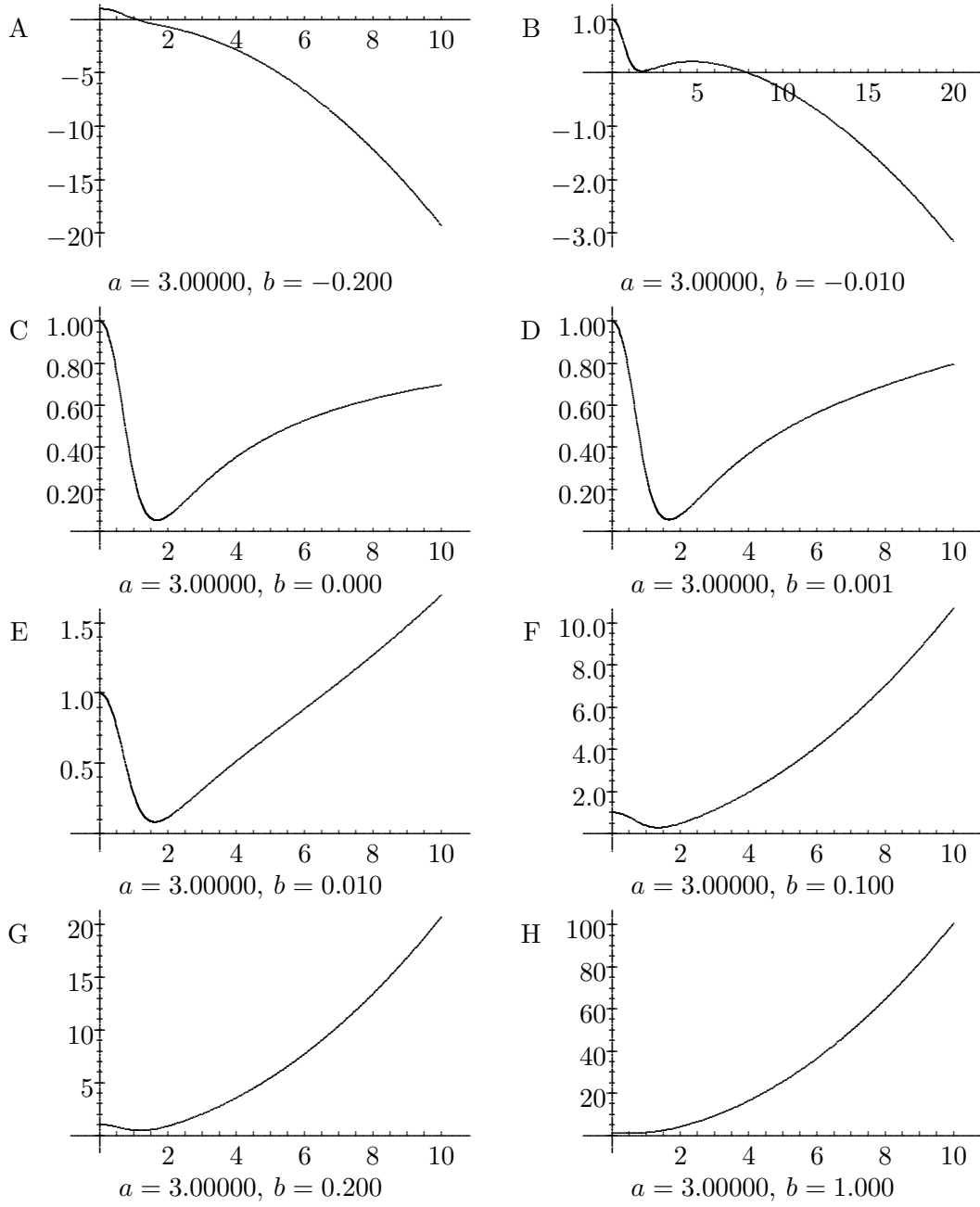


Fig. 2. Plots of the function $f(x)$ (Eq. (3.15)) for some values of parameters a and b (see the text for explanation)

Fig. 2A: $a = 3.00000, b = -0.200$; Fig. 2B: $a = 3.00000, b = -0.010$;
 Fig. 2C: $a = 3.00000, b = 0.000$; Fig. 2D: $a = 3.00000, b = 0.001$;
 Fig. 2E: $a = 3.00000, b = 0.010$; Fig. 2F: $a = 3.00000, b = 0.100$;
 Fig. 2G: $a = 3.00000, b = 0.200$; Fig. 2H: $a = 3.00000, b = 1.000$.

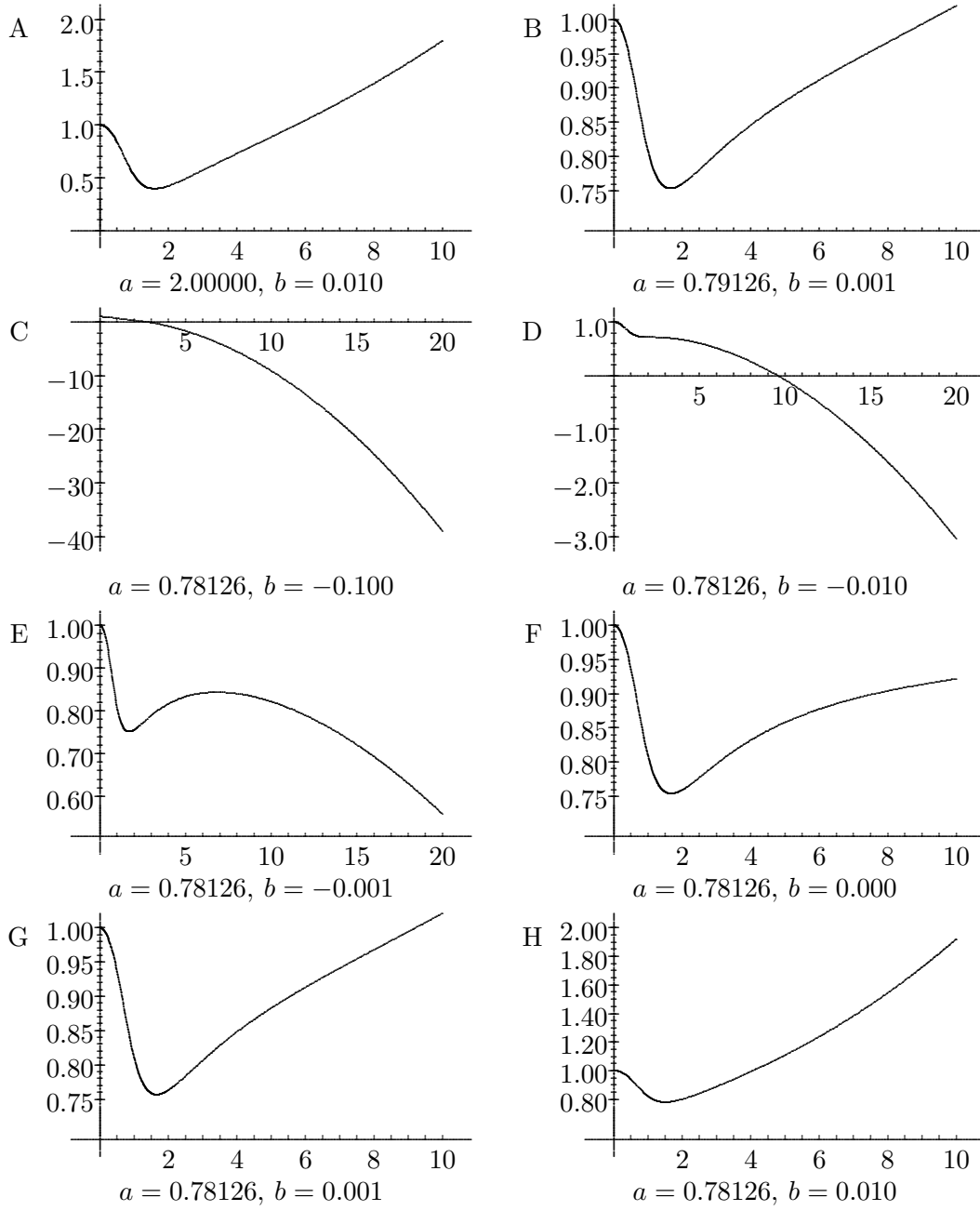


Fig. 3. Plots of the function $f(x)$ (Eq. (3.15)) for some values of parameters a and b (see the text for explanation)

Fig. 3A: $a = 2.00000$, $b = 0.010$; Fig. 3B: $a = 0.79126$, $b = 0.001$;
 Fig. 3C: $a = 0.78126$, $b = -0.100$; Fig. 3D: $a = 0.78126$, $b = -0.010$;
 Fig. 3E: $a = 0.78126$, $b = -0.001$; Fig. 3F: $a = 0.78126$, $b = 0.000$;
 Fig. 3G: $a = 0.78126$, $b = 0.001$; Fig. 3H: $a = 0.78126$, $b = 0.010$.

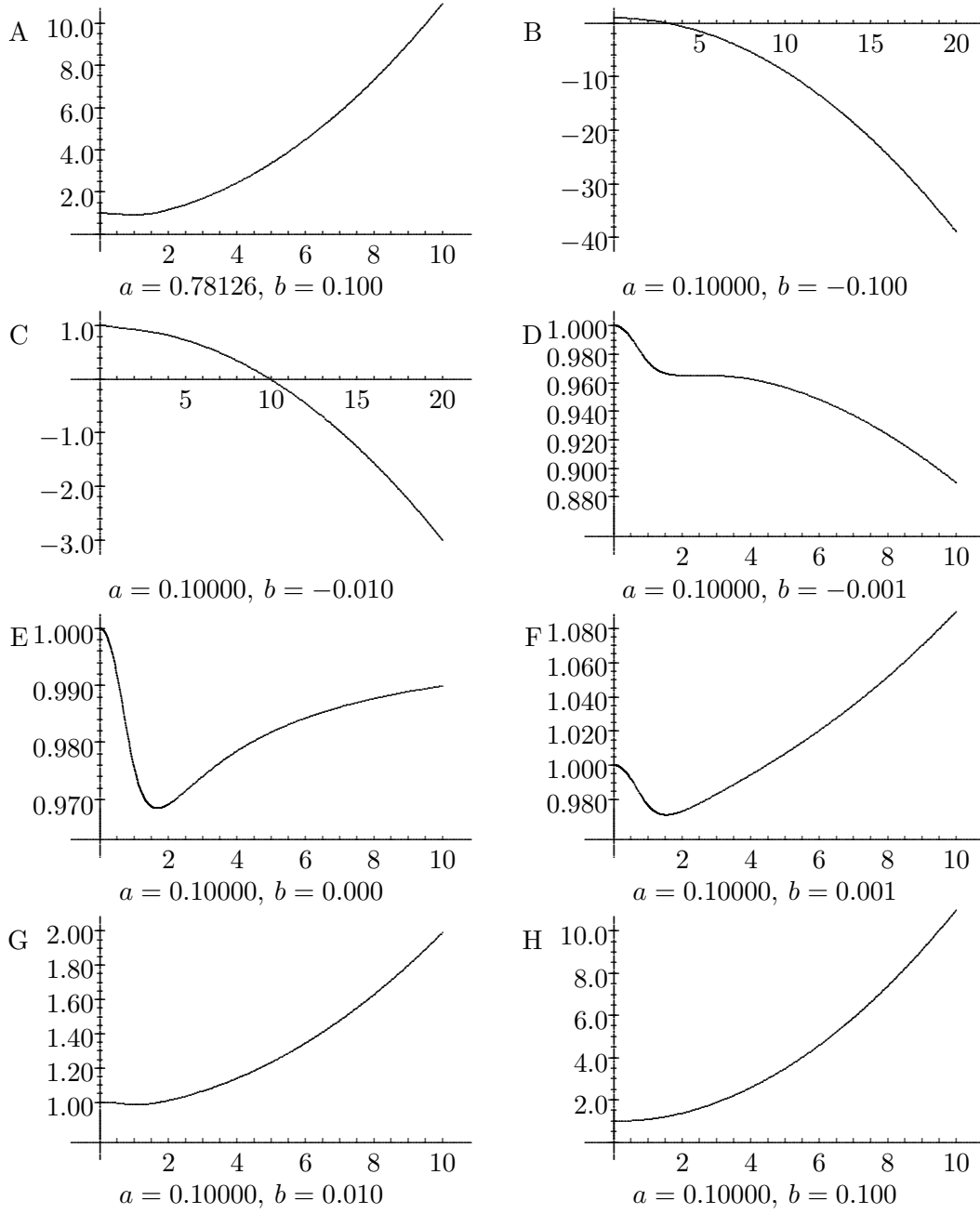


Fig. 4. Plots of the function $f(x)$ (Eq. (3.15)) for some values of parameters a and b (see the text for explanation)

Fig. 4A: $a = 0.78126, b = 0.100$; Fig. 4B: $a = 0.10000, b = -0.100$;
 Fig. 4C: $a = 0.10000, b = -0.010$; Fig. 4D: $a = 0.10000, b = -0.001$;
 Fig. 4E: $a = 0.10000, b = 0.000$; Fig. 4F: $a = 0.10000, b = 0.001$;
 Fig. 4G: $a = 0.10000, b = 0.010$; Fig. 4H: $a = 0.10000, b = 0.100$.

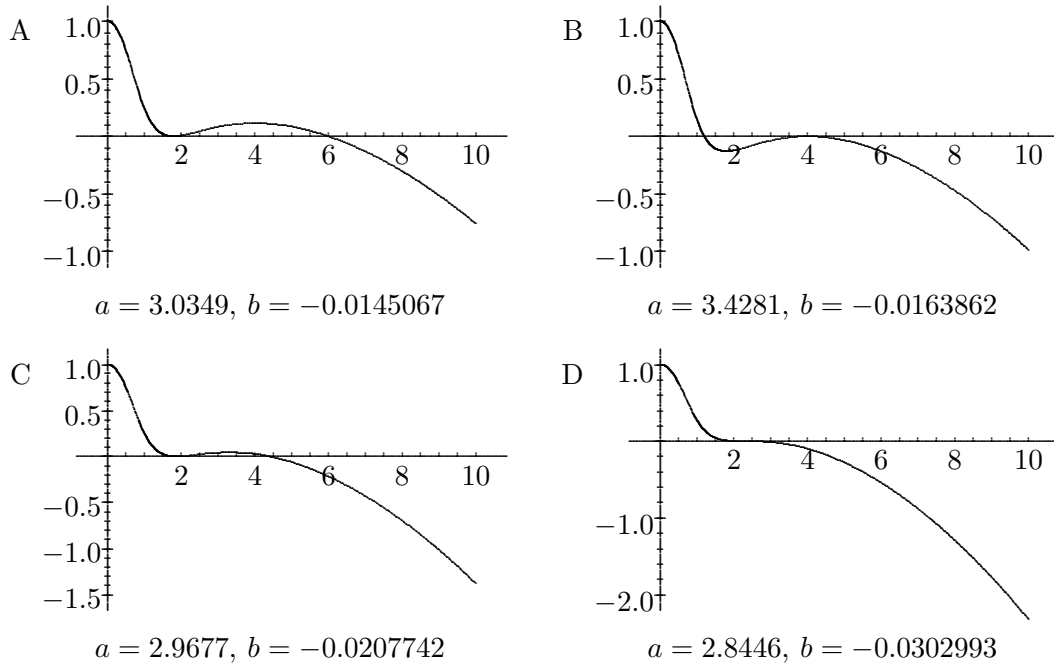


Fig. 5. Plots of the function $f(x)$ (Eq. (3.15) for some interesting values of a and b ($b < 0$), see the text for an explanation)

Fig. 5A: $a = 3.0349$, $b = -0.0145067$; Fig. 5B: $a = 3.4281$, $b = -0.0163862$;

Fig. 5C: $a = 2.9677$, $b = -0.0207742$; Fig. 5D: $a = 2.8446$, $b = -0.0302993$.

Eq. (3.17) has one solution for $b \geq 0$. However in the case of $b_0 < b < 0$ it has two solutions, one solution for $b = b_0$ and no solutions for $b < b_0$. Thus we can write $a_{\text{crt}} = a_{\text{crt}}(b)$, $x_{\text{crt}}(b)$ only for $b \geq 0$ (and $b = b_0$). In the remaining case $b_0 < b < 0$ we rewrite these equations as

$$b = a_{\text{crt}} \frac{x_{\text{crt}}^3 + g(x_{\text{crt}})(1 + x_{\text{crt}}^4)}{2x_{\text{crt}}^3(x_{\text{crt}}^4 + 1)} \quad (3.19)$$

$$a_{\text{crt}} = -\frac{2x_{\text{crt}}(x_{\text{crt}}^4 + 1)}{x_{\text{crt}}^3 + 3g(x_{\text{crt}})(x_{\text{crt}}^4 + 1)}.$$

The estimated value of b_0 is -0.0302993 . It seems that in the case of de Sitter horizon for $b < b_0 = -0.0302993$ there are not any horizon except the mentioned one.

In the case of $b < 0$ we have an interesting phenomenon due to the fact that the function f has two local extrema (one minimum and one maximum). For a special value of b_0 they collapse to one inflection point. This results in one horizon as a solution of a system of equations

$$\begin{aligned} f(x_{\text{crt}}, a_{\text{crt}}, b_0) &= 0 \\ \frac{\partial f}{\partial x}(x_{\text{crt}}, a_{\text{crt}}, b_0) &= 0 \\ \frac{\partial^2 f}{\partial x^2}f(x_{\text{crt}}, a_{\text{crt}}, b_0) &= 0 \end{aligned} \quad (3.20)$$

One finds $x_{\text{crt}} \approx 2.30331$, $a_{\text{crt}} \approx 2.8446$, $b_0 = -0.0302993$.

The points $x_{\text{crt}} \approx 2.30331$, $a_{\text{crt}} \approx 2.8446$ correspond to the minimum of the function $b(x)$ defined by the first equation (3.19).

In Fig. 6 we give plots for $x_{\text{crt}}(b)$ and $a_{\text{crt}}(b)$ for $b > 0$ and for $b < 0$. In the case of $b < 0$ we have two branches x_{crt} and a_{crt} .

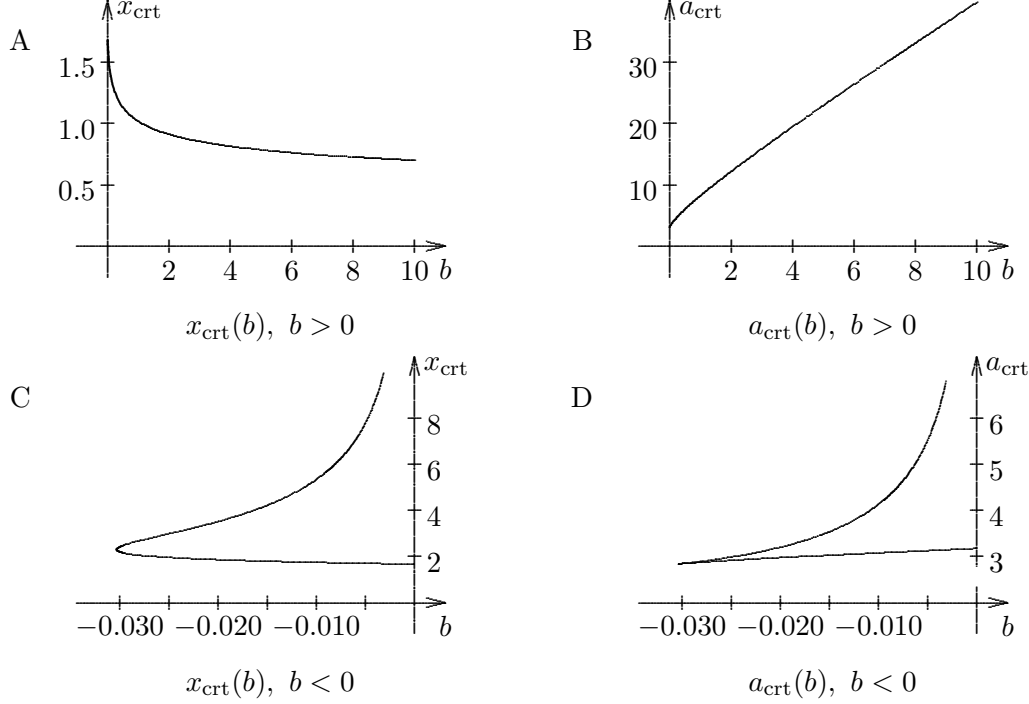


Fig. 6. Plots of $x_{\text{crt}}(b)$ and $a_{\text{crt}}(b)$ for $b > 0$ (Fig. 6A and Fig. 6B) and for $b < 0$ (Fig. 6C and Fig. 6D)

4 Dirac equation in the Nonsymmetric Kaluza–Klein Theory

In this section we deal with a generalization of a Dirac equation on P . Thus we consider spinor fields $\Psi, \bar{\Psi}$ on P transforming according to $\text{Spin}(1,4)$ (a double covering group of $\text{SO}(1,4)$ —de Sitter group). We want to couple these fields to gravity and electromagnetism. For Ψ and $\bar{\Psi}$ we have $\Psi, \bar{\Psi} : P \rightarrow \mathbb{C}^4$ and

$$\begin{aligned}\Psi(\varphi(g)p) &= \sigma(g^{-1})\Psi(p) \\ \bar{\Psi}(\varphi(g)p) &= \bar{\Psi}(p)\sigma(g),\end{aligned}\tag{4.1}$$

where $\sigma \in \mathcal{L}(\mathbb{C}^4)$, $p = (x, g_1) \in P$, $g, g_1 \in \text{U}(1)$.

On E we define spinor ordinary fields $\psi, \bar{\psi} : E \rightarrow \mathbb{C}^4$. We suppose that ψ and $\bar{\psi}$ are defined up to a phase factor and that

$$\begin{aligned}\psi^f(x) &= \Psi(f(x)) \\ \bar{\psi}^f(x) &= \bar{\Psi}(f(x))\end{aligned}\tag{4.2}$$

where $f : E \rightarrow P$ is a section of a bundle \underline{P} . In some sense spinor fields on P are lifts of spinors on E (see Appendix C),

$$\begin{aligned}\Psi(f(x)) &= \pi^*(\psi^f(x)), & \psi^f &= f^*\Psi \\ \bar{\Psi}(f(x)) &= \pi^*(\bar{\psi}^f(x)), & \bar{\psi}^f &= f^*\bar{\Psi}.\end{aligned}\tag{4.3}$$

Let us consider a different section of a bundle \underline{P} , $e : E \rightarrow P$. In this case we have

$$\begin{aligned}\psi^e &= e^*\Psi, & \bar{\psi} &= e^*\bar{\Psi}, & \psi^e(x) &= \Psi(e(x)), & \bar{\psi}^e(x) &= \bar{\Psi}(e(x)), \\ \psi^e(x) &= \psi^f(x) \exp\left(\frac{ikq}{\hbar c} \chi(x)\right), & \bar{\psi}^e(x) &= \bar{\psi}^f(x) \exp\left(-\frac{ikq}{\hbar c} \chi(x)\right),\end{aligned}$$

where kq is a charge of a fermion, $k = 0, \pm 1, \pm 2, \dots$, for an electron $k = 1$, χ is a gauge changing function.

Let us define an exterior gauge derivative $\overset{\text{gauge}}{d}$ of the field Ψ . One gets

$$d\Psi = \zeta_\mu \Psi \theta^\mu + \zeta_5 \Psi \theta^5 \tag{4.4}$$

and

$$\begin{aligned}\overset{\text{gauge}}{d} \Psi &= \text{hor } d\Psi = \zeta_\mu \Psi \theta^\mu \\ \overset{\text{gauge}}{d} \bar{\Psi} &= \text{hor } d\bar{\Psi} = \zeta_\mu \bar{\Psi} \theta^\mu.\end{aligned}\tag{4.5}$$

Let $\gamma_\mu \in \mathcal{L}(\mathbb{C}^4)$ be Dirac's matrices obeying the conventional relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{4.6}$$

(where $\eta_{\mu\nu}$ is a Minkowski tensor of signature $(- - - +)$) and let $B = B^+$ be a matrix such that

$$\gamma^{\mu+} = B\gamma^\mu B^{-1}, \quad \bar{\psi} = \psi^+ B \tag{4.7}$$

(the indices are raised by $\eta^{\mu\nu}$, an inverse tensor of $\eta_{\mu\nu}$), where “+” is a Hermitian conjugation, and

$$\sigma_{\mu\nu} = \frac{1}{8}[\gamma_\mu, \gamma_\nu]. \tag{4.8}$$

We define

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \in \mathcal{L}(\mathbb{C}^4).$$

One can easily check that

$$\{\gamma_A, \gamma_B\} = 2\bar{g}_{AB} \tag{4.9}$$

where

$$\begin{aligned}\bar{g}_{AB} &= \text{diag}(-1, -1, -1, +1, -1) \\ \text{and} \quad \gamma^A &= (\gamma^\alpha, \gamma^5)\end{aligned}\tag{4.10}$$

(the indices are raised by \bar{g}^{AB} , an inverse tensor of \bar{g}_{AB}). We have

$$\gamma^{5+} = B\gamma^5 B^{-1} \quad \text{and} \quad \bar{\Psi} = \Psi^+ B. \tag{4.11}$$

So

$$\gamma^{A+} = B\gamma^A B^{-1}. \tag{4.12}$$

On the manifold P we have an orthonormal coordinate system θ^A and we can perform an infinitesimal change of the frame

$$\begin{aligned}\theta^{A'} &= \theta^A + \delta\theta^A = \theta^A - \varepsilon^A_B \theta^B \\ \varepsilon_{AB} + \varepsilon_{BA} &= 0.\end{aligned}\tag{4.13}$$

If the spinor field Ψ corresponds to θ^A and Ψ' to $\theta^{A'}$ then we get

$$\begin{aligned}\Psi' &= \Psi + \delta\Psi = \Psi - \varepsilon^{AB} \hat{\sigma}_{AB} \Psi \\ \bar{\Psi}' &= \bar{\Psi} + \delta\bar{\Psi} = \bar{\Psi} + \bar{\Psi} \hat{\sigma}_{AB} \varepsilon^{AB}\end{aligned}\tag{4.14}$$

(Ψ and $\bar{\Psi}$ are Schouten σ -quantities (see Refs [33], [34]) where

$$\hat{\sigma}_{AB} = \frac{1}{8}[\gamma_A, \gamma_B].\tag{4.15}$$

Notice that the dimension of the spinor space for a $2n$ -dimensional space is 2^n and it is the same for a $(2n+1)$ -dimensional one (in our case $n=2$).

We take a spinor field for a 5-dimensional space P and assume that the dependence on the 5th dimension is trivial, i.e. Eq. (4.1) holds. Taking a section we obtain spinor fields on E .

Let us introduce some new notions. We introduce a Levi-Civita symbol and a dual Cartan's base

$$\bar{\eta}_{\alpha\beta\gamma\delta}, \quad \bar{\eta}_{1234} = \sqrt{-\det(g_{(\alpha\beta)})}\tag{4.16}$$

$$\bar{\eta}_\alpha = \frac{1}{2 \cdot 3} \bar{\theta}^\delta \wedge \bar{\theta}^\gamma \wedge \bar{\theta}^\beta \bar{\eta}_{\alpha\beta\gamma\delta}\tag{4.17}$$

$$\bar{\eta} = \frac{1}{4} \bar{\theta}^\alpha \wedge \bar{\eta}_\alpha.\tag{4.18}$$

We define

$$\begin{aligned}\eta_\alpha &= \pi^*(\bar{\eta}_\alpha) \\ \eta &= \pi^*(\bar{\eta})\end{aligned}\tag{4.19}$$

We rewrite here a Riemannian part of the connection (1.19) introducing the constant $\lambda = \frac{2\sqrt{G_N}}{c^2}$,

$$\tilde{w}^A_B = \left(\frac{\pi^*(\tilde{w}^\alpha_\beta) + \frac{\lambda}{2} \pi^*(F^\alpha_\beta) \theta^5}{-\frac{\lambda}{2} \pi^*(F_{\beta\gamma} \bar{\theta}^\gamma)} \middle| \frac{\frac{\lambda}{2} \pi^*(F^\alpha_\gamma \bar{\theta}^\gamma)}{0} \right)\tag{4.20}$$

(see Refs [35], [36]).

Let us consider exterior covariant derivatives of spinors Ψ and $\bar{\Psi}$,

$$\begin{aligned}\tilde{D}\Psi &= d\Psi + \tilde{w}^A_B \hat{\sigma}_A^B \Psi \\ \tilde{D}\bar{\Psi} &= d\bar{\Psi} - \tilde{w}^A_B \bar{\Psi} \hat{\sigma}_A^B\end{aligned}\tag{4.21}$$

with respect to the Riemannian connection \tilde{w}^A_B .

Now we introduce a derivative \mathcal{D} , i.e. an exterior “gauge” derivative of a new kind. This derivative may be treated as a generalization of minimal coupling scheme between spinor and electromagnetic field on P ,

$$\begin{aligned}\mathcal{D}\Psi &= \text{hor } D\Psi \\ \mathcal{D}\bar{\Psi} &= \text{hor } D\bar{\Psi}.\end{aligned}\tag{4.22}$$

We get

$$\begin{aligned}\mathcal{D}\Psi &= \widetilde{\mathcal{D}}\Psi - \frac{\lambda}{8} F^\alpha{}_\mu [\gamma_\alpha, \gamma_5] \Psi \theta^\mu \\ \mathcal{D}\bar{\Psi} &= \widetilde{\mathcal{D}}\bar{\Psi} + \frac{\lambda}{8} F^\alpha{}_\mu \bar{\Psi} [\gamma_\alpha, \gamma_5] \theta^\mu\end{aligned}\tag{4.23}$$

where

$$\begin{aligned}\widetilde{\mathcal{D}}\Psi &= \overset{\text{gauge}}{d}\Psi + \pi^*(\widetilde{w}^\alpha{}_\beta) \sigma_\alpha{}^\beta \Psi \\ \widetilde{\mathcal{D}}\bar{\Psi} &= \overset{\text{gauge}}{d}\bar{\Psi} - \pi^*(\widetilde{w}^\alpha{}_\beta) \bar{\Psi} \sigma_\alpha{}^\beta.\end{aligned}\tag{4.24}$$

The derivative $\widetilde{\mathcal{D}}$ is a covariant derivative with respect to both $\pi^*(\widetilde{w}^\alpha{}_\beta)$ and “gauge” at once. It introduces an interaction between electromagnetic and gravitational fields with Dirac’s spinor in a classical well-known way ($\widetilde{\mathcal{D}}\Psi = \text{hor } \widetilde{\mathcal{D}}\Psi$).

In Dirac theory we have the following Lagrangian for a spinor $\frac{1}{2}$ -spin field on E :

$$\mathcal{L}(\psi, \bar{\psi}, d) = i \frac{\hbar c}{2} (\bar{\psi} \bar{l} \wedge d\psi + d\bar{\psi} \wedge l\psi) + mc^2 \bar{\psi} \psi \bar{\eta}\tag{4.25}$$

where $\bar{l} = \gamma_\mu \bar{\eta}^\mu$.

Let us lift Lagrangian on a manifold P . We pass from spinors ψ and $\bar{\psi}$ to Ψ and $\bar{\Psi}$ and from the derivative d to $\overset{\text{gauge}}{d}$ or to $\widetilde{\mathcal{D}}$. This is a classical way. Moreover, we have to do with a theory which unifies gravity and electromagnetism and in order to get new physical effects we should pass to our new derivative \mathcal{D} . Simultaneously we pass from $\bar{\eta}$ to η and from \bar{l} to $\pi^*(\bar{l}) = l$.

In this way one gets

$$\mathcal{L}_D(\Psi, \bar{\Psi}, \mathcal{D}) = \frac{i\hbar c}{2} (\bar{\Psi} l \wedge \mathcal{D}\Psi + \mathcal{D}\bar{\Psi} \wedge l\Psi) + mc^2 \bar{\Psi} \Psi \eta.\tag{4.26}$$

Using formulae (4.23) one obtains

$$\mathcal{L}_D(\Psi, \bar{\Psi}, \mathcal{D}) = \mathcal{L}_D(\Psi, \bar{\Psi}, \widetilde{\mathcal{D}}) - i \frac{2\sqrt{G_N}}{c} \hbar F_{\mu\nu} \bar{\Psi} \gamma_5 \sigma^{\mu\nu} \Psi \eta\tag{4.27}$$

where

$$\mathcal{L}_D(\Psi, \bar{\Psi}, \widetilde{\mathcal{D}}) = \frac{i\hbar c}{2} (\bar{\Psi} l \wedge \widetilde{\mathcal{D}}\Psi + \widetilde{\mathcal{D}}\bar{\Psi} \wedge l\Psi) + mc^2 \bar{\Psi} \Psi \eta.\tag{4.28}$$

Now we should go back to a space-time E (see Appendix C) and we get the following Lagrangian

$$\mathcal{L}_D(\psi, \bar{\psi}, \mathcal{D}) = \mathcal{L}_D(\psi, \bar{\psi}, \widetilde{\mathcal{D}}) - i \frac{2\sqrt{G_N}}{c} \hbar F^{\mu\nu} \bar{\psi} \gamma_5 \sigma_{\mu\nu} \psi\tag{4.29}$$

$$\mathcal{L}_D(\psi, \bar{\psi}, \widetilde{\mathcal{D}}) = \frac{i\hbar c}{2} (\bar{\psi} \bar{l} \wedge \widetilde{\mathcal{D}}\psi + \widetilde{\mathcal{D}}\bar{\psi} \wedge l\psi) + mc^2 \bar{\psi} \psi \bar{\eta}.\tag{4.30}$$

We get a new term

$$-i \frac{2\sqrt{G_N}}{c} \hbar F^{\mu\nu} \bar{\psi} \gamma_5 \sigma_{\mu\nu} \psi.\tag{4.31}$$

It is an interaction of the electromagnetic field with an anomalous dipole electric moment. For such an anomalous interaction it reads

$$i \frac{d_{kk}}{2} F^{\mu\nu} \bar{\psi} \gamma_5 \sigma_{\mu\nu} \psi. \quad (4.32)$$

Our anomalous moment reads

$$d_{kk} = -\frac{4\sqrt{G_N}}{c} \hbar = -\frac{4l_{\text{pl}}}{\sqrt{\alpha}} q \simeq -7.56784835 \times 10^{-32} [\text{cm}] q \quad (4.33)$$

where l_{pl} is a Planck length

$$l_{\text{pl}} = \sqrt{\frac{\hbar G_N}{c^3}} \simeq 1.61199 \times 10^{-35} \text{m},$$

q is an elementary charge and

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$$

is a fine structure constant.

This term can be also rewritten in a different way,

$$-\frac{2}{\Lambda_p} (\hbar^3 c^5)^{1/2} F^{\mu\nu} \bar{\psi} \gamma_5 \sigma_{\mu\nu} \psi \quad (4.34)$$

where

$$\begin{aligned} \Lambda_p &= m_p c^2 \simeq 1.2209 \times 10^{19} \text{GeV} \\ m_p &= 2.1765 \times 10^{-8} \text{kg} \end{aligned} \quad (4.35)$$

are Planck energy scale and Planck mass. Thus we get a term which probably gives us a trace of New Physics on a Planck energy scale. This term is nonrenormalizable in Quantum Field Theory and it is of 5 order in mass units (i.e. $c = \hbar = 1$) divided by an energy (mass) scale.

The term (4.32) can be written in a very convenient way

$$d_{kk} \bar{\psi} (\beta (\vec{\Sigma} \cdot \vec{E} + i \vec{\alpha} \vec{B})) \psi \quad (4.36)$$

where

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \vec{\gamma} = \beta \vec{\alpha} \quad (4.37)$$

$$\vec{\Sigma} = -\gamma^5 \vec{\alpha} = \gamma^4 \gamma^5 \vec{\gamma} = \beta \gamma^5 \vec{\gamma} \quad (4.38)$$

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z), \quad (4.39)$$

I is the identity matrix 2×2 and $\vec{\sigma}$ are Pauli matrices. \vec{E} is an electric field and \vec{B} is a magnetic field. In this way our term introduces an anomalous dipole electric interaction and also an anomalous magnetic dipole interaction. Of course the magnetic interaction is negligible in comparison to ordinary magnetic moment interaction of an electron. One can easily calculate this anomalous magnetic moment of an electron in terms of Bohr magneton getting

$$\frac{4}{\sqrt{\alpha}} \left(\frac{m_e}{m_p} \right) \mu_B = 19.188 \times 10^{-21} \mu_B,$$

where m_e is a mass of an electron and $\mu_B = \frac{q\hbar}{2m_e}$ is a Bohr magneton. From the physical point of view the most important is the electric dipole moment (EDM). So we see that using spinors Ψ and $\bar{\Psi}$ and a derivative $\tilde{\mathcal{D}}$ in the Kaluza–Klein Theory we have achieved an additional gravitational-electromagnetic effect. It is just an existence of a dipole moment of a fermion, which value is determined by fundamental constants (only!). This is another “interference effect” between electromagnetic and gravitational fields in our unified field theory. Thirring also has achieved in his paper [37] a dipole electric moment of fermion of the same order. In his theory a minimal rest mass of fermion is of order of a Planck mass. Thus his theory cannot describe a fermion from the Standard Model. The anomalous moment in Thirring’s theory depends on a mass of a fermion. In order to get d_{kk} of order $10^{-32} [\text{cm}]q$ this must be of a Planck mass order. Otherwise the value of d_{kk} can be smaller. (In reality W. Thirring obtains two types of anomalous Pauli terms—electric and magnetic of the same order.)

In our case mass m may be arbitrary, e.g. $m = 0$. Thus we can consider also massless fermions. We can also consider chargeless fermions, i.e. for $k = 0$. It is also worth noticing that Thirring’s quantities Ψ and $\bar{\Psi}$ have nothing to do with our spinor fields Ψ and $\bar{\Psi}$ for a mysterious Thirring’s quantity φ which is absent in our theory (it appears also in Thirring’s definition of a parity operator). We develop the theory considered here also in ordinary Kaluza–Klein Theory and in the Kaluza–Klein theory with a torsion (see Refs [35], [38], [39]). Someone develops a theory using our spinors Ψ and $\bar{\Psi}$ getting also anomalous electric dipole moments (see Refs [40], [41]). We develop a similar approach for a Rarita–Schwinger field (see Ref. [42]). In the case of the Nonsymmetric Kaluza–Klein Theory we consider also a different approach (see [43], [44]). However now we consider the present as appropriate.

Let us consider operations of reflection defined on a manifold P . To define them we choose first a local coordinate system on P in such a way that we pass from θ^A to dx^A (see Section 1), i.e. $(\pi^*(dx^\alpha), dx^5)$. In this way

$$x^A = (x^\alpha, x^5), \quad x^\alpha = (\vec{x}, t). \quad (4.40)$$

Then

$$\Psi(p) = \Psi(x^A) = \Psi((\vec{x}, t), x^5) \quad (4.41)$$

and we define transformations: space reflection P (do not confuse with a manifold P), time reversal T , charge reflection C and combined transformations PC , $\theta = PCT$,

$$\Psi^C(x^\alpha, x^5) = C\Psi^*(x^\alpha, -x^5), \quad (4.42)$$

where $C^{-1}\gamma_\mu C = -\gamma_\mu^*$.

Taking a section f we get

$$(\psi^f)^C(x^\alpha) = C\psi^{f*}(x^\alpha) \quad (4.43)$$

and a charge changes the sign. The reflection $x^5 \rightarrow -x^5$ as a charge reflection has been already suggested by J. Rayski (see Ref. [45]). For the space coordinate reflection we have

$$\Psi^P(x^\alpha, x^5) = \gamma^4 \Psi(-\vec{x}, t, x^5). \quad (4.44)$$

Taking a section f we obtain

$$(\psi^f)^P(\vec{x}, t) = \gamma^4 \psi^f(-\vec{x}, t), \quad (4.45)$$

i.e. a normal parity operator on E .

This contrasts with Thirring's definition of the parity operator (Thirring was forced to change the definition of the parity operator on 5-dimensional space and he could not obtain a normal parity operator on E). The transformation of time-reversal T is defined by

$$\Psi^T(\vec{x}, t, x^5) = C^{-1} \gamma^1 \gamma^2 \gamma^3 \Psi^*(\vec{x}, -t, -x^5). \quad (4.46)$$

Taking a section f we get

$$(\psi^f)^T(\vec{x}, t) = C^{-1} \gamma^1 \gamma^2 \gamma^3 (\psi^f)^*(\vec{x}, -t) \quad (4.47)$$

and a charge does change sign, i.e. a normal time-reversal operator on a space-time.

To define a transformation $\theta = PCT$ we write

$$\Psi^\theta(\vec{x}, t, x^5) = -i \gamma^5 \Psi(-\vec{x}, -t, -x^5). \quad (4.48)$$

Taking a section f we get

$$(\psi^f)^\theta(\vec{x}, t) = -i \gamma^5 \psi^f(-\vec{x}, -t) \quad (4.49)$$

and a charge changes the sign. The transformation PC is as follows

$$\Psi^{PC}(\vec{x}, t, x^5) = \gamma^4 C \Psi^*(-\vec{x}, t, x^5). \quad (4.50)$$

Taking a section f we have

$$(\psi^f)^{PC}(\vec{x}, t) = \gamma^4 C (\psi^f)^*(-\vec{x}, t) \quad (4.51)$$

and a charge changes a sign.

It is clear now that the transformations obtained by us do not differ from those known from the literature.

The additional term in Lagrangian (4.27) breaks PC or T symmetries as in Thirring's theory (see Ref. [37]), but Thirring defines the operator PC in a different way. This can be easily seen by acting on both sides of Eq. (4.31) with the operator defined by Eq. (4.50). Of course this breaking is very weak and it cannot be linked to CP -breaking term in Cabbibo–Kobayashi–Maskawa matrix. From this breaking due to δ_{PC} -phase, which is responsible for PC nonconservation in K^0, \bar{K}^0 mesons decays and also for $D^0, \bar{D}^0, B_s, \bar{B}_s, B^0, \bar{B}^0$ and so on, see Ref. [46], we can get a dipole electric moment of an electron of order $8 \times 10^{-41} [\text{cm}]q$ (if there is not New Physics beyond SM, see Ref. [47]). This is because all Feynman diagrams which induce EDM of electron vanish to three loops order.

According to Ref. [47] electron EDM

$$d_e = \left(\frac{g_w^2}{32\pi^2} \right) \left(\frac{m_e}{M_w} \right) \left[\ln \frac{\Lambda^2}{M_w^2} + O(1) \right] d_W \quad (4.52)$$

where

$$d_W = J \left(\frac{g_W^2}{32\pi^2} \right) \left(\frac{q}{2M_W} \right) \frac{m_b^4 m_s^2 m_c^2}{M_W^2} \quad (4.53)$$

is EDM for a W boson, Λ is an energy scale for a New Physics (beyond SM),

$$J = s_1^2 s_2 s_3 c_1 c_2 c_3 \sin \delta_{CP} = 2.96 \times 10^{-5}$$

(see Ref. [48]) is a Jarlskog invariant, m_b, m_s, m_c are masses of quarks (we suppose the existence of three families of fermions in SM) and $s_i = \sin \theta_i$, $c_i = \cos \theta_i$, $i = 1, 2, 3$.

EDMs of an electron d_e and quarks can induce EDMs of paramagnetic and diamagnetic atoms

$$d_{\text{para}} \sim 10\alpha^2 Z^3 d_e \quad (4.54)$$

$$d_{\text{dia}} \sim 10Z^2 \left(\frac{R_N}{R_A}\right)^2 \tilde{d}_q. \quad (4.55)$$

For Thallium (Tl) and for Mercury (Hg) one gets

$$d_{\text{Tl}} = -585 d_e \quad (4.56)$$

$$d_{\text{Hg}} = 7 \times 10^{-3} e(\tilde{d}_u - \tilde{d}_d) + 10^{-2} d_e. \quad (4.57)$$

For a neutron

$$d_n = (1.4 \mp 0.6)(d_d - 0.25d_u) + (1.1 \pm 0.5)q(\tilde{d}_d + 0.5\tilde{d}_u)$$

where d_d, d_u are EDM of quarks and $\tilde{d}_d, \tilde{d}_u, \tilde{d}_q$ are color EDM operators (see Ref. [49] and references cited therein). Recently we have an upper bound on EDMs (see Ref. [50] and references cited therein)

$$|d_n| < 2.9 \times 10^{-26} [\text{cm}]q, \quad |d_e| < 1.6 \times 10^{-27} [\text{cm}]q, \quad d(^{199}\text{Hg}) < 3.1 \times 10^{-29} [\text{cm}]q.$$

In the case of θ -term in QCD we have also $d_n = 3 \times 10^{-16} \theta [\text{cm}]q$ (see Ref. [49]).

Recently there has been a significant progress in obtaining an upper limit on the EDM of an electron by using a polar molecule thorium monoxide (ThO). The authors of Ref. [51] obtained an upper limit on d_e ,

$$|d_e| < 8.7 \times 10^{-29} [\text{cm}]q. \quad (4.58)$$

This is only of three orders of magnitude bigger than our result (see Eq. (4.33)). From the other side there is also a progress in calculation of SM prediction of EDM for an electron coming from a phase δ_{CP} of CKM matrix. This calculation gives us the so called *equivalent* EDM (see Ref. [52]),

$$d_e^{\text{equiv}} \sim 10^{-38} [\text{cm}]q, \quad (4.59)$$

which is bigger of three orders of magnitude than the result from Ref. [46]. Moreover, still smaller of six orders than our result. The parameter θ from QCD is unknown and has no influence on EDM of an electron. The existence of EDM of an electron coming from Kaluza–Klein theory can help us in understanding of an asymmetry of matter-antimatter in the Universe. This EDM moment which breaks PC and T symmetry in an explicit way can have an influence on the surviving of an annihilation matter with antimatter following Big Bang.

It is interesting to notice that EDM from Kaluza–Klein Theory is the same for a muon (a μ meson) and a tauon (a τ meson) as for an electron. We get the same value for flavour states of neutrinos. Due to this, EDM of this value can influence oscillations of neutrinos species (see Ref. [53]).

To be honest, we write down a different, however trivial, coupling of spinor fields Ψ and $\bar{\Psi}$ in Kaluza–Klein. This is a coupling to a connection of the form

$$\hat{w}^A_B = \left(\frac{\pi^*(\tilde{w}^\alpha_\beta)}{0} \middle| \frac{0}{0} \right). \quad (4.60)$$

In this way Ψ and $\bar{\Psi}$ are transforming according to $\text{SL}(2, \mathbb{C})$ and new phenomena are absent, i.e. we have to do with Lagrangian (4.28).

Let us come back to neutrino oscillations in the presence of EDM. Let us write a Lagrangian for three neutrino species neglecting gravitational field:

$$\begin{aligned} \mathcal{L}_D(\Psi_\lambda, \bar{\Psi}_\lambda, d) = \sum_{\lambda=\alpha,\beta,\gamma} \left(\frac{i\hbar c}{2} (\bar{\Psi}_\lambda l \wedge d\Psi_\lambda + d\bar{\Psi}_\lambda \wedge l\Psi_\lambda) + i \frac{d_{kk}}{2} F^{\mu\nu} \bar{\Psi}_\lambda \gamma_5 \sigma_{\mu\nu} \Psi_\lambda \right) \\ + \sum_{\lambda,\lambda'=\alpha,\beta,\gamma} c^2 \bar{\Psi}_\lambda m_{\lambda\lambda'} \Psi_{\lambda'} \eta. \end{aligned} \quad (4.61)$$

Despite the smallness of d_{kk} its interaction with a strong electric and magnetic fields can result in sizeable effects (see Eq. (4.36)). $m_{\lambda\lambda'}$ is a mass matrix for neutrinos which is not diagonal. In particular $\alpha = e, \beta = \mu, \gamma = \tau$.

Let us consider mass eigenstates of our neutrinos $\Psi_a, a = 1, 2, 3$ (see [53])

$$\Psi_\lambda = \sum_{a=1,2,3} U_{\lambda a} \Psi_a. \quad (4.62)$$

The unitary matrix $U = (U_{\lambda a})$ diagonalizes the mass matrix $\bar{m} = (m_{\lambda\lambda'})$. The eigenvalues of the mass matrix are called $m_a, a = 1, 2, 3$.

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = U^+ \bar{m} U \quad (4.63)$$

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (4.64)$$

where $c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij}$, the angles $\theta_{ij} \in [0, \frac{\pi}{2}]$, $\delta \in [0, 2\pi]$ is a Dirac CP violation phase (see Refs. [46], [53], i means λ —flavour, j means a —mass eigenstate).

In the new spinor variables the Lagrangian (4.61) reads

$$\mathcal{L}_D(\Psi_a, \bar{\Psi}_a, d) = \sum_{a=1,2,3} \left(\frac{i\hbar c}{2} (\bar{\Psi}_a l \wedge d\Psi_a + d\bar{\Psi}_a \wedge l\Psi_a) + \bar{\Psi}_a M_a \Psi_a \eta \right), \quad (4.65)$$

where

$$M_a = m_a c^2 + i \frac{d_{kk}}{2} F^{\mu\nu} \gamma_5 \sigma_{\mu\nu} = m_a c^2 + d_{kk} \beta (\vec{\Sigma} \cdot \vec{E} + i \vec{\alpha} \cdot \vec{B}) \quad (4.66)$$

(see Eq. (4.36)).

Using initial conditions for mass eigenstates

$$\Psi_a(\vec{r}, t=0) = \Psi_a^{(0)}(\vec{r}) \quad (4.67)$$

$$\Psi_\lambda^{(0)}(\vec{r}) = U_{\lambda a} \Psi_a^{(0)}(\vec{r}) \quad (4.68)$$

$$\Psi_a^{(0)}(\vec{r}) = (U^{-1})_{a\lambda} \Psi_\lambda^{(0)}(\vec{r}) \quad (4.69)$$

we can solve an initial value problem for linear equations corresponding to the Lagrangian (4.65), finding an evolution in time of fields Ψ_a (they do not couple). Afterwards using (4.62) and (4.69) we find oscillations of three neutrino flavours under an influence of magnetic and electric fields due to additional term coming from Kaluza–Klein Theory. Field equations for Ψ_a (Euler–Lagrange equations for Lagrangian (4.65)) are given in the following Hamilton form

$$i\hbar c \frac{\partial \Psi_a}{\partial t} = H_a \Psi_a, \quad a = 1, 2, 3, \quad (4.70)$$

where

$$H_a = c \vec{\alpha} \cdot \vec{p} + \beta m_a c^2 - d_{kk} (\vec{\Sigma} \cdot \vec{E} + i \vec{\alpha} \cdot \vec{B}) \quad (4.71)$$

$$\vec{p} = -i\hbar \vec{\nabla}. \quad (4.72)$$

Thus eventually one gets

$$i\hbar c \frac{\partial \Psi_a}{\partial t} = -i\hbar c (\vec{\alpha} \cdot \vec{\nabla}) \Psi_a + m_a c^2 \beta \Psi_a - d_{kk} (\vec{\Sigma} \cdot \vec{E} + i \vec{\alpha} \cdot \vec{B}) \Psi_a, \quad a = 1, 2, 3. \quad (4.73)$$

Equations (4.73) are typical Dirac–Pauli equations. Moreover, they have a term which explicitly breaks PC transformation. We suppose $\vec{E} = \text{const}$, $\vec{B} = \text{const}$. For Eqs (4.73) are linear the general solutions are expressed by the Fourier integral

$$\begin{aligned} \Psi_a(\vec{r}, t) = & \int \frac{d^3 \vec{p}}{(2\pi)^{3/2}} e^{i \vec{p} \cdot \vec{r}} \\ & \times \sum_{\zeta=\pm 1} \left[a_a^{(\zeta)} u_a^{(\zeta)}(\vec{p}) \exp(-iE(+)_a^{(\zeta)} t) + b_a^{(\zeta)} v_a^{(\zeta)}(\vec{p}) \exp(-iE(-)_a^{(\zeta)} t) \right] \end{aligned} \quad (4.74)$$

where $a_a^{(\zeta)}, b_a^{(\zeta)}$ are arbitrary coefficients, $u_a^{(\zeta)}, v_a^{(\zeta)}$ are base spinors such that

$$H_a u_a^{(\zeta)} = E(+)_a^{(\zeta)} u_a^{(\zeta)} \quad (4.75)$$

$$H_a v_a^{(\zeta)} = E(-)_a^{(\zeta)} v_a^{(\zeta)}. \quad (4.76)$$

In the classical situation

$$E(+)_a^{(\zeta)} = -E(-)_a^{(\zeta)} \quad (4.77)$$

and $\zeta = \pm 1$ describes different polarization states of the fermions Ψ_a (see Refs [54], [55]). In our case $E(+)_a^{(+1)}, E(+)_a^{(-1)}, E(-)_a^{(-1)}, E(-)_a^{(+1)}$ are roots of the polynomial of the fourth order

$$\det(H_a(\vec{p}) - IE_a) = 0, \quad a = 1, 2, 3, \quad (4.78)$$

where I is the identity matrix 4×4 and

$$H_a(\vec{p}) = c\vec{\alpha} \cdot \vec{p} + \beta m_a c^2 - d_{kk}(\vec{\Sigma} \cdot \vec{E} + i\vec{\alpha} \cdot \vec{\beta}), \quad a = 1, 2, 3. \quad (4.79)$$

Spinors $u_a^{(\zeta)}, v_a^{(\zeta)}$ are eigenvectors corresponding to those eigenvalues. They are orthogonal. Using formulae (4.37)–(4.39) one transforms Eqs (4.78)–(4.79) into

$$H_a = \begin{pmatrix} m_a c^2 I - d_{kk}(\vec{E} \cdot \vec{\sigma}) & (c\vec{p} - id_{kk}\vec{B}) \cdot \vec{\sigma} \\ (c\vec{p} - id_{kk}\vec{B}) \cdot \vec{\sigma} & d_{kk}(\vec{E} \cdot \vec{\sigma}) - m_a c^2 I \end{pmatrix}, \quad a = 1, 2, 3, \quad (4.80)$$

and

$$\det \begin{pmatrix} (m_a c^2 - E_a)I - d_{kk}(\vec{E} \cdot \vec{\sigma}) & (c\vec{p} - id_{kk}\vec{B}) \cdot \vec{\sigma} \\ (c\vec{p} - id_{kk}\vec{B}) \cdot \vec{\sigma} & d_{kk}(\vec{E} \cdot \vec{\sigma}) - (m_a c^2 + E_a)I \end{pmatrix} = 0, \quad a = 1, 2, 3, \quad (4.81)$$

where I is the 2×2 identity matrix.

Using explicit forms of Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.82)$$

one eventually gets

$$\det \begin{pmatrix} m_a c^2 - E_a & -d_{kk}(E_x - iE_y) & cp_z - id_{kk}B_z & c(p_x - ip_y) \\ -d_{kk}E_z & & & -id_{kk}(B_x - iB_y) \\ -d_{kk}(E_x + iE_y) & m_a c^2 - E_a & c(p_x + ip_y) & -cp_z + id_{kk}B_z \\ & + d_{kk}E_z & -id_{kk}(B_x + iB_y) & \\ cp_z - id_{kk}B_z & c(p_x - ip_y) & -m_a c^2 - E_a & d_{kk}(E_x - iE_y) \\ & -id_{kk}(B_x - iB_y) & + d_{kk}E_z & \\ c(p_x + ip_y) & -cp_z + id_{kk}B_z & d_{kk}(E_x + iE_y) & -m_a c^2 - E_a \\ -id_{kk}(B_x + iB_y) & & & + d_{kk}E_z \end{pmatrix} = 0. \quad (4.83)$$

Using initial conditions we can determine coefficients $a_a^{(\zeta)}$ and $b_a^{(\zeta)}$, i.e. we expand $\Psi_a^{(0)}(\vec{r})$ into Fourier integral

$$\Psi_a^{(0)}(\vec{r}) = \int \frac{d^3 \vec{p}}{(2\pi)^{3/2}} e^{i\vec{p} \cdot \vec{r}} \sum_{\zeta=\pm 1} [a_a^{(\zeta)} u_a^{(\zeta)}(\vec{p}) + b_a^{(\zeta)} v_a^{(\zeta)}(\vec{p})], \quad a = 1, 2, 3. \quad (4.84)$$

We can consider several possibilities of neutrino flavour oscillations supposing e.g.

$$\Psi_\alpha^{(0)}(\vec{r}) = \xi(\vec{r}) \quad \text{and} \quad \Psi_\beta^{(0)}(\vec{r}) = \Psi_\gamma^{(0)}(\vec{r}) = 0. \quad (4.85)$$

In this way

$$\Psi_a^{(0)}(\vec{r}) = U_{a\alpha} \xi(\vec{r}) \quad (4.86)$$

which can be considered as initial conditions for oscillations.

Moreover, this problem is beyond the scope of this paper and will be considered elsewhere.

Let us notice that our generalization of a minimal coupling scheme Eq. (4.22) induces a new connection on P .

$$\check{w}^A{}_B = \text{hor}(\tilde{w}^A{}_B) \quad (4.87)$$

$$\text{or } \check{w}^A{}_B = \left(\begin{array}{c|c} \pi^*(\tilde{w}^{\alpha}{}_{\beta}) & \frac{\lambda}{2}\pi^*(F^{\alpha}{}_{\gamma}\bar{\theta}^{\gamma}) \\ \hline -\frac{\lambda}{2}\pi^*(F_{\beta\gamma}\bar{\theta}^{\gamma}) & 0 \end{array} \right). \quad (4.88)$$

This connection is metric but with non-vanishing torsion. Properties of this connection have been extensively examined (also in the case of nonabelian Kaluza–Klein Theories) in Ref. [56].

Let us consider the following problem. What would it mean for Physics if someone measured an EDM for an electron of the value $d_{kk} = -\frac{4l_{\text{pl}}}{\sqrt{\alpha}}q$ as predicted in this section? It would mean the fifth dimension is a reality in the sense of a 5-dimensional Minkowski space.

An experiment which measures such a quantity strongly supports an idea of rotations around the fifth axis in this space (the fifth dimension is a space-like). This EDM exists only due to these rotations. Otherwise spinor fields couple to a connection (4.60) and there is not a new effect.

Even P is a 5-dimensional manifold, the additional fifth dimension is not necessarily of the same nature as the remaining four dimensions, in particular three space dimensions. This dimension is a gauge dimension connected to the electromagnetic field. Moreover, we can develop this theory using Yang–Mills’ fields and also Higgs’ fields using dimensional reduction procedure, expecting some additional effects. It means we can expect something as “travelling” along additional dimensions. This perspective would have a tremendous importance for Physics and Technology.

Simultaneously an existence of an EDM of an electron has also very great impact on our understanding of PC and T symmetries breaking. This is also very important.

Thus a mentioned measurement with an answer: *Yes*, would have very important physical, technological and even philosophical implications.

Conclusions

In the paper we consider four problems:

1. Charge confinement in the Nonsymmetric Kaluza–Klein Theory.
2. Gravito-electromagnetic waves solutions in this theory.
3. An influence of a cosmological constant on a spherically-symmetric static solution.
4. Dirac equations in Nonsymmetric Kaluza–Klein Theory.

There are some further prospects:

1. To find similar conditions for confinement (of colour) in a nonabelian version in the theory.
2. To find similar gravito–Yang–Mills waves.
3. To find spherical and cylindrical waves in the theory.

Finally, we give some remarks. There are some misunderstandings connecting Kaluza–Klein Theory, Einstein’s Unified Field Theory, Nonsymmetric Gravitation Theory (NGT), Nonsymmetric Kaluza–Klein Theory (NKKT), Nonsymmetric Jordan–Thiry Theory (NJTT).

1. First of all we comment a constant $\lambda = \frac{2\sqrt{G_N}}{c^2}$. The constant λ appeared as a free parameter in this theory. Moreover in order to get Einstein equations with electromagnetic sources known from GR it is fixed and it is not free any more. Why is there not a Planck’s length? I explain it shortly. The Kaluza theory is classical for a paper published by him is classical as a classical paper in the scientific literature. It is also classical for this theory is not quantum. For this we cannot get here a Planck’s constant. This is simply for we need a Planck’s constant in order to construct the Planck’s length. Planck’s constant is absent in Kaluza theory for this theory is classical (non-quantum). The Planck’s length appeared in the further development done by O. Klein. O. Klein considered a Klein–Gordon equation in 5-dimensional extension. The Planck’s constant is present in Klein–Gordon equation. This equation can be considered as an equation for a classical scalar field. In Kaluza–Klein theory Planck’s length appears as a scale of length.

2. The classical Kaluza theory as a realistic unified field theory has been abandoned by 1950’s. Moreover, due to some mathematical investigations a deep structure has been discovered behind the theory. Let us describe it shortly. First of all it happens that behind Maxwell theory of electromagnetism there is a principal fibre bundle over a space-time with a structural group $U(1)$ and a connection defined on this bundle is an electromagnetic field. Gauge transformation, four-potential, the first pair of Maxwell equation obtained a clear geometrical meaning in terms of a fibre bundle approach (see Section 1 and Appendix A).

It happens also that a classical Kaluza theory is a theory of metrized (in a natural way) electromagnetic fibre bundle (see Ref. [57]).

This is a true unification of the two fundamental principles of invariance in physics: a gauge invariance principle and a coordinate invariance principle.

In Section 2 of Ref. [35] a classical KKT in this setting has been described (see also the last two lines of page 576 with a fixing of the constant λ).

Moreover this paper is devoted to the KKT with torsion in such a way that we put in the place of GR the Einstein–Cartan theory obtaining new features the so-called “interference effects” between gravity and electromagnetism going to some effects which are small, moreover in principle measurable in experiment. The Nonsymmetric Kaluza–Klein Theory has been constructed using ideas and mathematical formalism similar to those from Ref. [35], i.e. to Kaluza–Klein Theory with torsion.

3. Let us consider Einstein Unified Field Theory. A. Einstein started this theory in 1920’s. In 1950 he came back to this theory describing it in Appendix II of the fifth edition of his famous book *The Meaning of Relativity* (see Ref. [58]).

It is worth to mention that there are many versions of this theory. The oldest Einstein–Thomas theory and after that Einstein–Strauss theory, Einstein–Kaufmann theory. There are also two approaches, weak and strong field equations. The Einstein Unified Field Theory can be also considered as a real theory and Hermitian theory. A slight deviation is the so-called Bonnor’s Unified Field Theory. In all of these approaches there are two fundamental notions: nonsymmetric affine connection $\Gamma^\lambda_{\mu\nu} \neq \Gamma^\lambda_{\nu\mu}$ and the nonsymmetric metric $g_{\mu\nu} \neq g_{\nu\mu}$. Connection and metric can be real or Hermitian. In this theory there is also a second connection $W^\lambda_{\mu\nu} \neq W^\lambda_{\nu\mu}$. Connection $\Gamma^\lambda_{\mu\nu}$ is a so-called constrained connection, $W^\lambda_{\mu\nu}$ is called unconstrained. All

of these approaches have no free parameters. Some parameters which appear in solutions of field equations are *integration constants*.

What was an aim to construct such theories? The aim was to find a unified theory of gravity and electromagnetism in such a way that GR and Maxwell theory appear as some limit of the theory. This approach ended with fiasco. It was impossible to obtain a Lorentz force. It was impossible to obtain a Coulomb law too.

One can find all references to all versions of Einstein Unified Field Theory in Refs [1], [3], [4], [5] and we will not quote them here. Moreover, it is worth to mention that A. Einstein considered this theory as a theory of an extended gravitation. Moreover, there is a reference of A. Einstein's idea to treat this theory as a theory of an extended gravity only. A. Einstein published a paper on it in *Scientific American* (the only one Einstein's paper in this journal, see Ref. [59]).

Geometrical–mathematical properties of Einstein Unified Field Theory have been described in a book by Vaclav Hlavatý (see Ref. [60]).

In those times A. Einstein started a program of geometrization of physics. Some notions of this program have been described in Ref. [61].

There is also an approach to this theory going in a different direction. It has been summarized in the book by A. H. Klotz (see Ref. [62]).

4. Let us comment NGT (Nonsymmetric Gravitational Theory) by J. W. Moffat. J. W. Moffat reinterpreted Einstein Unified Field Theory as a theory of a pure gravitational field (see Ref. [7]). He introduced material sources to the formalism. Moreover, he introduced in his theory an additional universal constant. He and his co-workers developed this idea getting many interesting results which are in principle testable by astronomical observations in the Solar System and beyond. He was using both real and Hermitian theory. Simultaneously he developed a formalism with two connections $\Gamma^\lambda_{\mu\nu}$ and $W^\lambda_{\mu\nu}$.

5. Let us comment the Nonsymmetric Kaluza–Klein and the Nonsymmetric Jordan–Thiry Theory. I posed and developed these theories using the nonsymmetric metrization of an electromagnetic fibre bundle using differential forms formalism as in Ref. [35].

Early results concerning the Nonsymmetric Kaluza–Klein Theory have been published in Refs [63], [64].

The final result of the theory with some developments has been published in Ref. [4]. The paper contains also an extension to the Nonsymmetric Jordan–Thiry Theory with a scalar field Ψ (or ρ). In order to get a pure Nonsymmetric Kaluza–Klein Theory it is enough to put $\Psi = 0$ (or $\rho = 1$). All new features as some “interference effects” between electromagnetic fields and gravitation have been quoted in Introduction of Ref. [4]. The theory has no free parameters except integration constants in solutions.

It is possible to get an extension of the theory to the non-Abelian case. In this case we have one free parameter. Moreover, this parameter can be fixed by a cosmological constant. The final version of this theory can be found in Ref. [3]. In Ref. [1] one can find also an extension to the case with Higgs' field and spontaneous symmetry breaking. In the last case there are three free parameters which can be fixed by a cosmological constant and scales of masses.

I do not refer in my paper to the paper Ref. [65], for the authors are using completely different approach (it is better to say *three approaches*). This approach is far away from investigations in my work. Moreover, in future both approaches can meet and we will shake hands. The only one point which is now common is a starting point, a classical Kaluza Theory. We do not refer to Ref. [66].

This paper deals with some problems in NGT. However, NGT considered by them has only a little touch with NGT considered here. They introduced a mass for skew-symmetric field $B_{\mu\nu} = -B_{\nu\mu}$ (in our notation it is $g_{[\mu\nu]}$). Moreover, the $g_{[\mu\nu]}$ can obtain a mass in a linear approximation of Nonsymmetric Non-Abelian Kaluza–Klein Theory due to a cosmological constant and it is not necessary to introduce a mass term. It seems that this is a completely different approach (see Ref. [66]). For a cure of NGT by a cosmological constant see also Ref. [67].

Let us notice the following fact. Einstein’s Unified Field Theory has been abandoned as a realistic unified theory for it has been proved using EIH (Einstein–Infeld–Hoffman) method that there is not a Lorentz force term and Coulomb like law.

These are disadvantages of Einstein Unified Field Theory but not NGT. This works now for our advantage, for we do not see any term like Lorentz force and Coulomb-like law in gravitational physics (I do not mean a Newton gravitational law which can be obtained in Einstein Unified Field Theory). Someone said: “*it is clever to use advantages, moreover, more clever is to use disadvantages*” and this is a case. Moreover, in the Nonsymmetric Kaluza–Klein Theory we get Lorentz force term from $(N+4)$ -dimensional (5-dimensional in an electromagnetic case) geodetic equations (see Refs [1], [2], [3], [4]).

All additional notions in the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory have been described in [1]. We get from $(N+4)$ -dimensional theory ($N = n + n_1$) four-dimensional equations due to an invariance of a nonsymmetric metric and a connection with respect to the right action of the group (in the electromagnetic case this is a biinvariance of the action of a group $U(1)$).

Let us notice also the following fact. Equations obtained in the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory are different from these in pure NGT. Due to this we can obtain nonsingular solutions of field equations in the electromagnetic case. These solutions possess a nonsingular metric $g_{(\alpha\beta)}$ and nonsingular electric field. The asymptotic behaviour is as in the case of Reissner–Nordström solution (see Refs [1], [4] and Section 3). This is impossible to get in pure NGT.

6. In Section 2 we consider gravito-electromagnetic waves. They have nothing to do with gravito-electromagnetism in General Relativity. The notion of a wave is not so easy as in Halliday’s textbook. It is more general, see Ref. [68]. A wave carries an information. Roughly speaking, it can be modulated. It means, it should possess an arbitrary function of e.g. $(z - t)$. In the case of nonlinear waves we use Riemann invariants (see e.g. Ref. [69]) and the wave possesses an arbitrary function of a Riemann invariant. Moreover, a gravitational wave is more complicated, e.g. there is not a plane gravitational wave. The gravitational field of such a wave is zero (the curvature tensor induced by a metric describing a plane wave is zero). Moreover, we can consider generalized plane waves (see Ref. [70]). Gravitational waves considered in [70] are not only gravitational waves in a linear approximation. There are here exact solutions of Einstein equations which can describe a very strong gravitational field. So we consider in Section 2 wave solutions of Nonsymmetric Kaluza–Klein Theory in the sense of the mentioned definition of a wave. Field equations in the Nonsymmetric Kaluza–Klein Theory describe gravitational and electromagnetic fields. The solutions depend on arbitrary functions of $(z - t)$. In the limit of zero skewon and zero electromagnetic fields we get generalized plane waves known from the book by Zakharov. Thus those solutions are gravito-electromagnetic waves. We use some achievements in Einstein Unified Field Theory as results in a pure gravity (see Refs [22]–[25]). The electromagnetic wave has remarkable properties to have both invariants of an electromagnetic field $S = F_{\mu\nu}F^{\mu\nu}$, $P = F_{\mu\nu}F^{*\mu\nu}$ equal to zero. The electromagnetic field for gravito-electromagnetic field wave has this property.

Appendix A

In the appendix we describe the notation and definitions of geometric quantities used in the paper. We use a smooth principal bundle which is an ordered sequence

$$\underline{P} = (P, F, G, E, \pi), \quad (\text{A.1})$$

where P is a total bundle manifold, F is typical fibre, G , a Lie group, is a structural group, E is a base manifold and π is a projection. In our case $G = \text{U}(1)$, E is a space-time, $\pi : P \rightarrow E$. We have a map $\varphi : P \times G \rightarrow P$ defining an action of G on P . Let $a, b \in G$ and ε be a unit element of the group G , then $\varphi(a) \circ \varphi(b) = \varphi(ba)$, $\varphi(\varepsilon) = \text{id}$, where $\varphi(a)p = \varphi(p, a)$. Moreover, $\pi \circ \varphi(a) = \pi$. For any open set $U \subset E$ we have a local trivialization $U \times G \simeq \pi^{-1}(U)$. For any $x \in E$, $\pi^{-1}(\{x\}) = F_x \simeq G$, F_x is a fibre over x and is equal to F . In our case we suppose $G = F$, i.e. a Lie group G is a typical fibre. ω is a 1-form of connection on P with values in the algebra of G , \mathfrak{G} . In the case of $G = \text{U}(1)$ we use a notation α (an electromagnetic connection). Lie algebra of $\text{U}(1)$ is R . Let $\varphi'(a)$ be a tangent map to $\varphi(a)$ whereas $\varphi^*(a)$ is the contragradient to $\varphi'(a)$ at a point a . The form ω is a form of ad-type, i.e.

$$\varphi^*(a)\omega = \text{ad}'_{a^{-1}}\omega, \quad (\text{A.2})$$

where $\text{ad}'_{a^{-1}}$ is a tangent map to the internal automorphism of the group G

$$\text{ad}_a(b) = aba^{-1}. \quad (\text{A.3})$$

In the case of $\text{U}(1)$ (abelian) the condition (A.2) means

$$\mathcal{L}_{\zeta_5} \alpha = 0, \quad (\text{A.4})$$

where ζ_5 is a Killing vector corresponding to one generator of the group $\text{U}(1)$. Thus this is a vector tangent to the operation of the group $\text{U}(1)$ on P , i.e. to $\varphi_{\exp(i\chi)}$, $\chi = \chi(x)$, $x \in E$, \mathcal{L} is a Lie derivative along ζ_5 . We may introduce the distribution (field) of linear elements H_r , $r \in P$, where $H_r \subset T_r(P)$ is a subspace of the space tangent to P at a point r and

$$v \in H_r \iff \omega_r(v) = 0. \quad (\text{A.5})$$

So

$$T_r(P) = V_r \oplus H_r, \quad (\text{A.6})$$

where H_r is called a subspace of *horizontal* vectors and V_r of *vertical* vectors. For vertical vectors $v \in V_r$ we have $\pi'(v) = 0$. This means that v is tangent to the fibres.

Let

$$v = \text{hor}(v) + \text{ver}(v), \quad \text{hor}(v) \in H, \quad \text{ver}(v) \in V_r. \quad (\text{A.7})$$

It is proved that the distribution H_r is equal to choosing a connection ω . We use the operation hor for forms, i.e.

$$(\text{hor } \beta)(X, Y) = \beta(\text{hor } X, \text{hor } Y), \quad (\text{A.8})$$

where $X, Y \in T(P)$.

The 2-form of a curvature is defined as follows

$$\Omega = \text{hor } d\omega = D\omega, \quad (\text{A.9})$$

where D means an exterior covariant derivative with respect to ω . This form is also of ad-type.

For Ω the structural Cartan equation is valid

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega], \quad (\text{A.10})$$

where

$$[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]. \quad (\text{A.11})$$

Bianchi's identity for ω is as follows

$$D\Omega = \text{hor } d\Omega = 0. \quad (\text{A.12})$$

The map $f : E \supset U \rightarrow P$ such that $f \circ \pi = \text{id}$ is called a *section* (U is an open set).

From physical point of view it means choosing a gauge. A covariant derivative on P is defined as follows

$$D\Psi = \text{hor } d\Psi. \quad (\text{A.13})$$

This derivative is called a *gauge derivative*. Ψ can be a spinor field on P .

In this paper we use also a linear connection on manifolds E and P , using the formalism of differential forms. So the basic quantity is a one-form of the connection ω^A_B . The 2-form of curvature is as follows

$$\Omega^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B \quad (\text{A.14})$$

and the two-form of torsion is

$$\Theta^A = D\theta^A, \quad (\text{A.15})$$

where θ^A are basic forms and D means exterior covariant derivative with respect to connection ω^A_B . The following relations are established connections with generally met symbols

$$\begin{aligned} \omega^A_B &= \Gamma^A_{BC} \theta^C \\ \Theta^A &= \frac{1}{2} Q^A_{BC} \theta^B \wedge \theta^C \\ Q^A_{BC} &= \Gamma^A_{BC} - \Gamma^A_{CB} \\ \Omega^A_B &= \frac{1}{2} R^A_{BCD} \theta^C \wedge \theta^D, \end{aligned} \quad (\text{A.16})$$

where Γ^A_{BC} are coefficients of connection (they do not have to be symmetric in indices B and C), R^A_{BCD} is a tensor of a curvature, Q^A_{BC} is a tensor of a torsion in a holonomic frame. Covariant exterior derivation with respect to ω^A_B is given by the formula

$$\begin{aligned} D\Xi^A &= d\Xi^A + \omega^A_C \wedge \Xi^C \\ D\Sigma^A_B &= d\Sigma^A_B + \omega^A_C \wedge \Sigma^C_B - \omega^C_B \wedge \Sigma^A_C. \end{aligned} \quad (\text{A.17})$$

The forms of a curvature Ω^A_B and torsion Θ^A obey Bianchi's identities

$$\begin{aligned} D\Omega^A_B &= 0 \\ D\Theta^A &= \Omega^A_B \wedge \theta^B. \end{aligned} \quad (\text{A.18})$$

All quantities introduced here can be found in Ref. [71].

In this paper we use a formalism of a fibre bundle over a space-time E with an electromagnetic connection α and traditional formalism of differential geometry for linear connections on E and P . In order to simplify the notation we do not use fibre bundle formalism of frames over E and P . A vocabulary connected geometrical quantities and gauge fields (Yang–Mills fields) can be found in Ref. [57].

In Ref. [72] we have also a similar vocabulary (see Table I, Translation of terminology). Moreover, we consider a little different terminology. First of all we distinguished between a gauge potential and a connection on a fibre bundle. In our terminology a gauge potential $A_\mu \bar{\theta}^\mu$ is in a particular gauge e (a section of a bundle), i.e.

$$A_\mu \bar{\theta}^\mu = e^* \omega \quad (\text{A.19})$$

where $A_\mu \bar{\theta}^\mu$ is a 1-form defined on E with values in a Lie algebra \mathfrak{G} of G . In the case of a strength of a gauge field we have similarly

$$\frac{1}{2} F_{\mu\nu} \bar{\theta}^\mu \wedge \bar{\theta}^\nu = e^* \Omega \quad (\text{A.20})$$

where $F_{\mu\nu} \bar{\theta}^\mu \wedge \bar{\theta}^\nu$ is a 2-form defined on E with values in a Lie algebra \mathfrak{G} of G .

Using generators of a Lie algebra \mathfrak{G} of G we get

$$A = A_\mu^a \bar{\theta}^\mu X_a = e^* \omega \quad \text{and} \quad F = \frac{1}{2} F_{\mu\nu}^a \bar{\theta}^\mu \wedge \bar{\theta}^\nu X_a = e^* \Omega \quad (\text{A.21})$$

where

$$[X_a, X_b] = C_{ab}^c X_c, \quad a, b, c = 1, 2, \dots, n, \quad n = \dim G (= \dim \mathfrak{G}), \quad (\text{A.22})$$

are generators of \mathfrak{G} , C_{ab}^c are structure constants of a Lie algebra of G , \mathfrak{G} , $[\cdot, \cdot]$ is a commutator of Lie algebra elements.

In this paper we are using Latin lower case letters for 3-dimensional space indices. Here we are using Latin lower case letters as Lie algebra indices. It does not result in any misunderstanding.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + C_{bc}^a A_\mu^b A_\nu^c. \quad (\text{A.23})$$

In the case of an electromagnetic connection α the field strength F does not depend on gauge (i.e. on a section of a bundle).

Finally it is convenient to connect our approach using gauge potentials A_μ^a with usually met (see Ref. [73]) matrix valued gauge quantities A_μ and $F_{\mu\nu}$. It is easy to see how to do it if we consider Lie algebra generators X_a as matrices. Usually one supposes that X_a are matrices of an adjoint representation of a Lie algebra \mathfrak{G} , T^a with a normalization condition

$$\text{Tr}(\{T^a, T^b\}) = 2\delta^{ab}, \quad (\text{A.24})$$

where $\{\cdot, \cdot\}$ means anticommutator in an adjoint representation.

In this way

$$A_\mu = A_\mu^a T^a, \quad (\text{A.25})$$

$$F_{\mu\nu} = F_{\mu\nu}^a T^a. \quad (\text{A.26})$$

One can easily see that if we take

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (\text{A.27})$$

from Ref. [73] we get

$$F_{\mu\nu} = (F^a_{\mu\nu})T^a, \quad (\text{A.28})$$

where $F^a_{\mu\nu}$ is given by (A.23). From the other side if we take a section f , $f : U \rightarrow P$, $U \subset E$, and corresponding to it

$$\bar{A} = \bar{A}^a_\mu \bar{\theta}^\mu X_a = f^* \omega \quad (\text{A.29})$$

$$\bar{F} = \frac{1}{2} \bar{F}^a_{\mu\nu} \bar{\theta}^\mu \wedge \bar{\theta}^\nu X_a = f^* \Omega \quad (\text{A.30})$$

and consider both sections e and f we get transformation from A^a_μ to \bar{A}^a_μ and from $F^a_{\mu\nu}$ to $\bar{F}^a_{\mu\nu}$ in the following way. For every $x \in U \subset E$ there is an element $g(x) \in G$ such that

$$f(x) = e(x)g(x) = \varphi(e(x), g(x)). \quad (\text{A.31})$$

Due to (A.2) one gets

$$\bar{A}(x) = \text{ad}'_{g^{-1}(x)} A(x) + g^{-1}(x) dg(x) \quad (\text{A.32})$$

$$\bar{F}(x) = \text{ad}'_{g^{-1}(x)} F(x) \quad (\text{A.33})$$

where $\bar{A}(x), \bar{F}(x)$ are defined by (A.29)–(A.30) and $A(x), F(x)$ by (A.21). The formulae (A.32)–(A.33) give a geometrical meaning of a gauge transformation (see Ref. [57]). In an electromagnetic case $G = \text{U}(1)$ we have similarly, if we change a local section from e to f we get

$$f(x) = \varphi(e(x), \exp(i\chi(x))) \quad (f : U \supset E \rightarrow P)$$

and $\bar{A} = A + d\chi$.

Moreover, in the traditional approach (see Ref. [73]) one gets

$$\bar{A}_\mu(x) = U(x)^{-1} A_\mu(x) U(x) + U^{-1}(x) \partial_\mu U(x) \quad (\text{A.34})$$

$$\bar{F}_{\mu\nu}(x) = U^{-1}(x) F_{\mu\nu}(x) U(x), \quad (\text{A.35})$$

where $U(x)$ is the matrix of an adjoint representation of a Lie group G .

For an action of a group G on P is via (A.2), $g(x)$ is exactly a matrix of an adjoint representation of G . In this way (A.32)–(A.33) and (A.34)–(A.35) are equivalent.

Let us notice that usually a Lagrangian of a gauge field (Yang–Mills field) is written as

$$\mathcal{L}_{\text{YM}} \sim \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \quad (\text{A.36})$$

where $F_{\mu\nu}$ is given by (A.26)–(A.27). It is easy to see that one gets

$$\mathcal{L}_{\text{YM}} \sim h_{ab} F^a_{\mu\nu} F^{b\mu\nu} \quad (\text{A.37})$$

where

$$h_{ab} = C^d_{ac} C^c_{bd} \quad (\text{A.38})$$

is a Cartan–Killing tensor for a Lie algebra \mathfrak{G} , if we remember that X_a in adjoint representation are given by structure constants C_{ab}^c .

Moreover, in Refs [1, 3] we use the notation

$$\Omega = \frac{1}{2} H_{\mu\nu}^a \theta^\mu \wedge \theta^\nu X_a. \quad (\text{A.39})$$

In this language

$$\mathcal{L}_{\text{YM}} = \frac{1}{8\pi} h_{ab} H_{\mu\nu}^a H^{\mu\nu b}. \quad (\text{A.40})$$

It is easy to see that

$$e^*(H_{\mu\nu}^a \theta^\mu \wedge \theta^\nu X_a) = F_{\mu\nu}^a \bar{\theta}^\mu \wedge \bar{\theta}^\nu X_a. \quad (\text{A.41})$$

Thus (A.40) is equivalent to (A.37) and to (A.36). (A.36) is invariant to a change of a gauge. (A.40) is invariant with respect to the action of a group G on P .

Let us notice that $h_{ab} F_{\mu\nu}^a F^{b\mu\nu} = h_{ab} H_{\mu\nu}^a H^{b\mu\nu}$, even $H_{\mu\nu}^a$ is defined on P and $F_{\mu\nu}^a$ on E . In the non-abelian case it is more natural to use $H_{\mu\nu}^a$ in place of $F_{\mu\nu}^a$.

Appendix B

In this appendix we find a formula for $H_{\nu\mu}$ from Eq. (1.48). In order to do this let us solve this equation perturbatively. According to Ref. [4] and Ref. [74] one gets

$$H_{\alpha\beta} = H_{\alpha\beta}^{(0)} + \delta H_{\alpha\beta}^{(1)} + \delta H_{\alpha\beta}^{(2)} + \dots \quad (\text{B.1})$$

where $H_{\alpha\beta}^{(0)}$ is $H_{\alpha\beta}$ in zero order of expansion with respect to $h_{\alpha\beta}$ where

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} = \eta_{\alpha\beta} + h_{(\alpha\beta)} + h_{[\alpha\beta]} = \eta_{\alpha\beta} + h_{(\alpha\beta)} + g_{[\alpha\beta]} \quad (\text{B.2})$$

and $\delta H_{\alpha\beta}^{(k)}$ is a k -th correction to $H_{\alpha\beta}^{(0)}$. One gets

$$g^{\mu\sigma} g_{\nu\sigma} = (\eta^{\mu\sigma} + \delta h^{\mu\sigma} + \delta h^{\mu\sigma} + \dots)(\eta_{\nu\sigma} + h_{\nu\sigma}) = \delta_\nu^\mu. \quad (\text{B.3})$$

From (B.3) one gets

$$\delta h^{\mu\nu} = -\eta^{\mu\sigma} \eta^{\nu\beta} h_{\beta\sigma} \quad (\text{B.4})$$

$$\delta h^{\mu\nu} = -\eta^{\nu\beta} \delta h^{\mu\sigma} h_{\beta\sigma} = \eta^{\nu\beta} \eta^{\mu\gamma} \eta^{\alpha\sigma} h_{\sigma\gamma} h_{\beta\alpha} \quad (\text{B.5})$$

where $\delta h^{\mu\nu}$ are k -th corrections to $\eta^{\mu\nu}$ (a zero order of an inverse tensor of $g_{\mu\nu}$, $\eta^{\mu\nu}$ is an inverse Minkowski tensor). We get

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\sigma} \eta^{\nu\beta} h_{\beta\sigma} + \eta^{\mu\gamma} \eta^{\nu\beta} \eta^{\alpha\sigma} h_{\beta\alpha} h_{\sigma\gamma}. \quad (\text{B.6})$$

Eq. (1.20) can be rewritten in a more convenient form

$$H_{\beta\sigma} - g^{\gamma\delta} [g_{[\beta\delta]} H_{\gamma\sigma} + g_{[\gamma\sigma]} H_{\beta\delta}] = F_{\beta\sigma} - 2g_{[\delta\sigma]} g^{\delta\gamma} F_{\beta\gamma}. \quad (\text{B.7})$$

Using Eq. (B.7) and writing

$$H_{\beta\alpha} = A_{\alpha\beta} + B_{\alpha\beta} \quad (\text{B.8})$$

$$A_{\alpha\beta} = A_{\beta\alpha}, \quad B_{\alpha\beta} = -B_{\beta\alpha} \quad (\text{B.9})$$

we can easily prove that $A_{\alpha\beta} = 0$. This means that $H_{\alpha\beta} = -H_{\beta\alpha}$ if $F_{\alpha\beta} = -F_{\beta\alpha}$. Using Eqs (B.6) and (B.7) one gets

$$\begin{aligned} H_{\alpha\beta}^{(0)} &= F_{\alpha\beta} \\ \delta H_{\beta\sigma}^{(1)} &= \eta^{\gamma\delta} (h_{[\beta\delta]} F_{\gamma\sigma} - h_{[\alpha\delta]} F_{\gamma\beta}) \\ \delta H_{\beta\sigma}^{(2)} &= \eta^{\gamma\delta} \eta^{\rho\alpha} (h_{(\rho\gamma)} (h_{[\sigma\delta]} F_{\alpha\beta} - h_{[\beta\delta]} F_{\sigma\alpha})) \end{aligned} \quad (\text{B.10})$$

and eventually

$$H_{\beta\alpha}^{(2)} = F_{\beta\alpha} + (\eta^{\gamma\delta} - h^{(\gamma\delta)}) (h_{[\beta\delta]} F_{\gamma\alpha} - h_{[\alpha\delta]} F_{\gamma\beta}), \quad (\text{B.11})$$

where

$$h^{(\gamma\delta)} = \eta^{\alpha\delta} \eta^{\beta\gamma} h_{(\alpha\beta)}. \quad (\text{B.12})$$

Eq. (B.11) can be rewritten in the form

$$H_{\nu\mu}^{(2)} = F_{\nu\mu} - \tilde{g}^{(1)(\tau\alpha)} (g_{[\mu\tau]} F_{\alpha\nu} - g_{[\nu\tau]} F_{\alpha\mu}) \quad (\text{B.13})$$

where $\tilde{g}^{(1)(\tau\alpha)}$ is an inverse tensor for $g_{(\alpha\beta)}$ up to the first order of expansion with respect to $h_{(\alpha\beta)}$. One can easily generalize this equation to any order k getting

$$H_{\nu\mu}^{(k)} = F_{\nu\mu} - \tilde{g}^{(k-1)(\tau\alpha)} (g_{[\mu\tau]} F_{\alpha\nu} - g_{[\nu\tau]} F_{\alpha\mu}). \quad (\text{B.14})$$

Taking $k \rightarrow \infty$ we get formula (1.48), where $H_{\mu\nu}^{(\infty)} = H_{\mu\nu}$, $\tilde{g}_{(\alpha\beta)}^{(\infty)} = \tilde{g}_{(\alpha\beta)}$.

Let us get Eq. (1.48) from a general formula in n -dimensional generalization of Einstein Unified Field Theory obtained by Hlavatý and Wrede (see Refs [60] and [75]). One gets

$$\begin{aligned} \Gamma_{WM}^N &= \tilde{\Gamma}_{WM}^N + \frac{1}{2} (K_{WM}^N - 2k_{[M}^A K_{W]AB} k^{NB}) \\ &\quad + h^{NE} \{ K_{E(W}^A k_{M)A} + k_{C}^B [k_{(M}^C K_{W)AB} k_{E}^A - K_{EAB} k_{(W}^A k_{M)C}] \} \end{aligned} \quad (\text{B.15})$$

where

$$\gamma_{AB} = h_{AB} + k_{AB} \quad (\text{B.16})$$

$$h_{AB} = h_{BA}, \quad k_{AB} = -k_{BA} \quad (\text{B.17})$$

$$K_{ABC} = -\tilde{\nabla}_A k_{BC} - \tilde{\nabla}_B k_{CA} + \tilde{\nabla}_C k_{AB}, \quad (\text{B.18})$$

$\tilde{\Gamma}_{WM}^N$ is a Levi-Civita connection generated by $h_{AB} = \gamma_{(AB)}$ ($\gamma_{[AB]} = k_{AB}$). $\tilde{\nabla}_A$ is a covariant derivative with respect to the connection $\tilde{\Gamma}_{WM}^N$.

The connection Γ^N_{WM} is a solution of the equation

$$D\gamma_{A+B-} = D\gamma_{AB} - \gamma_{AD}Q^D_{BC}(\Gamma)\theta^C = 0, \quad A, B, C, D, N, M = 1, 2, \dots, n, \quad (\text{B.19})$$

where D is an exterior covariant derivative with respect to a connection Γ .

$$h^{AB}h_{BC} = \delta^A_C \quad (\text{B.20})$$

and all indices are raised by h^{AB} (E. Schrödinger was surprised that it was possible to find a solution to (B.19) in a covariant form). The formula (B.15) is more general than that from Refs [60, 75] for in Eq. (B.15) $\tilde{\Gamma}^N_{WM}$ are coefficients of a Levi-Civita connection. This connection can be considered in nonholonomic frame. Thus $\tilde{\Gamma}^N_{WM}$ can be nonsymmetric in indices W and M . In Refs [60, 75] $\tilde{\Gamma}^N_{WM}$ mean Christoffel symbols. Moreover, the proof is exactly the same as in Refs [60, 75]. The authors of [60, 75] are using a natural nonholonomic frame connected to the nonsymmetric tensor γ_{AB} in order to find formula (B.15). Moreover, this nonholonomic frame has nothing to do with the frame we consider. They are supposing $\det(\gamma_{AB}) \neq 0$ and $\det(\gamma_{(AB)}) \neq 0$, which is equivalent to our assumptions (1.5) and (1.6) (in the case $n = 5$ for γ_{AB} given by Eq. (1.15)). Let us notice there is not any constraint imposed on a torsion of the connection.

V. Hlavatý and C. R. Wrede were first to consider n -dimensional generalization of a geometry from Einstein Unified Field Theory with nonsymmetric real tensor γ_{AB} .

Here we are using capital Latin indices as indices of many-dimensional manifolds. This does not result in any misunderstanding. In general, in non-Abelian theory, even in the case with spontaneous symmetry breaking, $n = 4 + N + N_1$, where N is the dimension of a Lie group and N_1 is the dimension of a homogeneous space (see Refs [1, 3, 5]).

In our case we have $n = 5$ and γ_{AB} is given by Eq. (1.15). It is easy to see that

$$\Gamma^5_{\mu\nu} = H_{\mu\nu} \quad (\text{B.21})$$

(in a lift horizontal basis, unholonomic frame). Thus it is enough to calculate $\Gamma^5_{\mu\nu}$. One gets

$$\Gamma^5_{\omega\mu} = \tilde{\Gamma}^5_{\omega\mu} - \frac{1}{2}K_{\omega\mu 5} - \frac{1}{4}\{K_{5\omega}{}^\alpha k_{\mu\alpha} + K_{5\mu}{}^\alpha k_{\omega\alpha} - k_{\gamma}{}^\beta k_{\alpha\beta} k_{\omega}{}^\alpha k_{\mu}{}^\gamma - k_{\gamma}{}^\beta K_{5\alpha\beta} k_{\mu}{}^\alpha k_{\omega}{}^\gamma\} \quad (\text{B.22})$$

where all indices are raised by $h^{\alpha\beta}$

$$h^{\alpha\beta}h_{\alpha\gamma} = \delta^\beta_\gamma \quad (\text{B.23})$$

$$g_{\alpha\beta} = h_{\alpha\beta} + k_{\alpha\beta} \quad (\text{B.24})$$

$$h_{\alpha\beta} = g_{(\alpha\beta)}, \quad k_{\alpha\beta} = g_{[\alpha\beta]}.$$

We are keeping notation from Refs [60, 75].

Moreover in Ref. [35] (see Eq. (2.16)) Levi-Civita connection coefficients for $\tilde{\Gamma}^N_{WN}$ generated by $\gamma_{(AB)} = h_{AB}$ are calculated. One has $\tilde{\Gamma}^\alpha_{\beta 5} = F^\alpha_\beta$, $\tilde{\Gamma}^5_{\beta\gamma} = F_{\beta\gamma}$, $\tilde{\Gamma}^\beta_{5\gamma} = F^\beta_\gamma$, $\tilde{\Gamma}^\alpha_{\beta\gamma} = \tilde{\tilde{\Gamma}}^\alpha_{\beta\gamma}$ for $\lambda = 2$ ($n = 5$ in Kaluza-Klein Theory). It is easy to see that they are not symmetric in indices (they are not Christoffel symbols for a frame is not holonomic, it is a lift horizontal basis). The remaining coefficients are zero.

Using these results one gets

$$\tilde{\nabla}_\mu k_{\mu 5} = -F^\tau_\omega k_{\mu\tau} = -\tilde{\nabla}_\omega k_{5\mu} \quad (\text{B.25})$$

$$\tilde{\nabla}_5 k_{\omega\mu} = -F^\tau{}_\mu k_{\omega\tau} - F^\tau{}_\omega k_{\tau\mu} \quad (\text{B.26})$$

$$\tilde{\nabla}_\mu k_{\tau\omega} = \tilde{\tilde{\nabla}}_\mu k_{\tau\omega} \quad (\text{B.27})$$

where we use the fact that

$$k_{\mu 5} = k_{55} = k_{5\mu} = 0 \quad (\text{B.28})$$

$$\partial_5 k_{\omega\mu} = 0. \quad (\text{B.29})$$

Eventually we get

$$K_{\omega\mu 5} = 2(F^\tau{}_\omega k_{\mu\tau} - F^\tau{}_\mu k_{\omega\tau}) \quad (\text{B.30})$$

$$K_{5\alpha\beta} = 2(F^\tau{}_\omega k_{\mu\tau} - F^\tau{}_\mu k_{\tau\omega}) \quad (\text{B.31})$$

$$K_{\omega 5\mu} = 2F^\tau{}_\mu k_{\tau\omega}. \quad (\text{B.32})$$

The remaining K_{ABC} are zero. One gets

$$H_{\omega\mu} = F_{\omega\mu} - F^\tau{}_\omega k_{\mu\tau} + F^\tau{}_\mu k_{\omega\tau}. \quad (\text{B.33})$$

Coming back to our notation from the paper (i.e. $k_{\mu\nu} = g_{[\mu\nu]}$, $h^{\tau\alpha} = \tilde{g}^{(\tau\alpha)}$) we get

$$H_{\omega\mu} = F_{\omega\mu} - \tilde{g}^{(\tau\alpha)} F_{\alpha\omega} g_{[\mu\tau]} + \tilde{g}^{(\tau\alpha)} F_{\alpha\mu} g_{[\omega\tau]}, \quad (\text{B.34})$$

i.e. Eq. (1.48). In this way we have a consistency in our theory getting the same results from both methods.

Appendix C

In this paper we consider two kinds of spinor fields $\Psi, \bar{\Psi}$ and $\psi, \bar{\psi}$ defined respectively on P and E . Spinor fields Ψ and $\bar{\Psi}$ transform according to $\text{Spin}(1, 4)$ and $\psi, \bar{\psi}$ according to $\text{Spin}(1, 3) \simeq \text{SL}(2, \mathbb{C})$. We have

$$U(g)\Psi(X) = D^F(g)\Psi(g^{-1}X), \quad X \in M^{(1,4)}, \quad g \in \text{SO}(1, 4). \quad (\text{C.1})$$

$\text{SO}(1, 4)$ acts linearly in $M^{(1,4)}$ (5-dimensional Minkowski space). The Lorentz group $\text{SO}(1, 3) \subset \text{SO}(1, 4)$. D^F is a representation of $\text{SO}(1, 4)$ (de Sitter group) such that after a restriction to its subgroup $\text{SO}(1, 3)$ we get

$$D^F|_{\text{SO}(1,3)}(\Lambda) = L(\Lambda), \quad (\text{C.2})$$

where

$$L(\Lambda) = D^{(1/2,0)}(\Lambda) \oplus D^{(0,1/2)}(\Lambda) \quad (\text{C.3})$$

is a Dirac representation of $\text{SO}(1, 3)$. More precisely, we deal with representations of $\text{Spin}(1, 4)$ and $\text{Spin}(1, 3) \simeq \text{SL}(2, \mathbb{C})$ (see Ref. [76]). In other words, we want spinor fields Ψ and $\bar{\Psi}$ to transform according to such a representation of $\text{Spin}(1, 4)$ which is induced by a Dirac representation of $\text{SL}(2, \mathbb{C})$. The complex dimensions of both representations are the same: 4. The same are also Clifford algebras

$$C(1, 4) \simeq C(1, 3) \quad (\text{C.4})$$

(see Refs [77], [78]).

One gets (up to a phase)

$$\Psi|_{\text{SL}(2,\mathbb{C})} = \psi. \quad (\text{C.5})$$

Spinor fields ψ and $\bar{\psi}$ transform according to Dirac representation, $\bar{\psi} = \psi^\dagger B$. Our matrices γ_μ and γ_A are representations of $C(1, 3)$ ($C(1, 4)$). One can consider projective representations for Ψ and ψ , i.e. representations of $\text{Spin}(1, 3) \otimes \text{U}(1)$ and $\text{SL}(2, \mathbb{C}) \otimes \text{U}(1)$. Moreover, we do not develop this idea here.

In this paper we develop the following approach to spinor fields on E and on P . We introduce orthonormal frames on E (dx^1, dx^2, dx^3, dx^4) and on P ($dX^1 = \pi^*(dx^1), dX^2 = \pi^*(dx^2), dX^3 = \pi^*(dx^3), dX^4 = \pi^*(dx^4), dX^5$). Our spinors Ψ on $(P, \gamma_{(AB)})$ and ψ on $(E, g_{(\alpha\beta)})$ are defined as complex bundles \mathbb{C}^4 over P or E with homomorphisms $\rho : C(1, 4) \rightarrow \mathcal{L}(\mathbb{C}^4)$ (resp. $\rho : C(1, 3) \rightarrow \mathcal{L}(\mathbb{C}^4)$) of bundles of algebras over P (resp. E) such that for every $p \in P$ (resp. $x \in E$), the restriction of ρ to the fiber over p (resp. x) is equivalent to spinor representation of a Clifford algebra $C(1, 4)$ (resp. $C(1, 3)$), i.e. D^F (resp. Dirac representation, see Refs [79], [80]). (There is also a paper on a similar subject (see Ref. [81]).) Spinor fields Ψ and ψ are sections of these bundles. There is also an approach to consider spinor bundles for Ψ and ψ as bundles associated to principal bundles of orthonormal frames for $(P, \gamma_{(AB)})$ or $(E, g_{(\alpha\beta)})$ (spin frames). Spinor fields Ψ and ψ are sections of these bundles. In our case we consider spinor fields Ψ and $\bar{\Psi}$ transforming according to (4.13) and (4.14). In the case of ψ and $\bar{\psi}$ we have

$$\begin{aligned} \bar{\theta}^{\alpha'} &= \bar{\theta}^\alpha + \delta\bar{\theta}^\alpha = \bar{\theta}^\alpha - \varepsilon^\alpha_\beta \bar{\theta}^\beta \\ \bar{\varepsilon}_{\alpha\beta} + \bar{\varepsilon}_{\beta\alpha} &= 0. \end{aligned} \quad (\text{C.6})$$

If the spinor field ψ corresponds to $\bar{\theta}^\alpha$ and ψ' to $\bar{\theta}^{\alpha'}$ we get

$$\begin{aligned} \psi' &= \psi + \delta\psi = \psi - \bar{\varepsilon}^{\alpha\beta} \sigma_{\alpha\beta} \psi \\ \bar{\psi}' &= \bar{\psi} + \delta\bar{\psi} = \bar{\psi} + \bar{\psi} \bar{\varepsilon}^{\alpha\beta} \sigma_{\alpha\beta}. \end{aligned} \quad (\text{C.7})$$

Spinor fields Ψ and $\bar{\Psi}$ are ψ and $\bar{\psi}$ in any section of a bundle P . Simultaneously we suppose conditions (4.2).

Similarly as for $\Psi, \bar{\Psi}$ one gets

$$\begin{aligned} \tilde{D}\psi &= d\psi + \tilde{w}^\alpha_\beta \sigma_\alpha^\beta \psi \\ \tilde{D}\bar{\psi} &= d\bar{\psi} - \tilde{w}^\alpha_\beta \bar{\psi} \sigma_\alpha^\beta \end{aligned} \quad (\text{C.8})$$

$$\begin{aligned} \tilde{\mathcal{D}}\psi &= \text{hor } \tilde{D}\psi = \overset{\text{gauge}}{d} \psi + \tilde{w}^\alpha_\beta \sigma_\alpha^\beta \psi \\ \tilde{\mathcal{D}}\bar{\psi} &= \text{hor } \tilde{D}\bar{\psi} = \overset{\text{gauge}}{d} \bar{\psi} - \tilde{w}^\alpha_\beta \bar{\psi} \sigma_\alpha^\beta. \end{aligned} \quad (\text{C.9})$$

Appendix D

In this paper we proceed a unification and geometrization of gravitational and electromagnetic interactions. Moreover, there is an approach (see Refs [82], [83], [84], [85], [86], [87]) which is going in a different direction. In that direction, the authors of those papers are transforming all

possible alternative theories of gravitation described by some geometric notions, i.e. connections, torsions, metric tensors and also with nonstandard Lagrangians, i.e. nonlinear Lagrangians ($f(R)$, $f(R^{\mu\nu}R_{\mu\nu})$, $f((R^\alpha_{\beta\mu\nu}\varepsilon^{\mu\nu\lambda\rho}R^\beta_{\alpha\lambda\rho})^2)$ etc.) to GR with additional “matter fields”. In our notation f means an arbitrary function, R is a scalar curvature, $R_{\mu\nu}$ is a Ricci tensor, $R^\alpha_{\beta\mu\nu}$ means a curvature tensor (no necessary Riemann–Christoffel tensor). One can consider also some different Lagrangians, i.e. $f(g_{\mu\nu}, R_{\mu\nu})$. In this way some unified theories, even Einstein Unified Field Theory, can be transformed into GR (General Relativity) plus some additional “matter fields”, i.e. scalar fields, vector fields and so on. This is possible of course by using Legendre transformation techniques to define a new metric (symmetric) tensor and a Levi-Civita connection compatible with this tensor.

The interpretation of this new tensor and a new connection can be complex. They could not have clear physical interpretation. In the case of Nonsymmetric Kaluza–Klein Theory considered in this paper we proceed in a little different way.

Let us consider our nonsymmetric connection \overline{W}^λ_μ and \overline{w}^λ_μ on a space-time E . Using results from Ref. [60] we can write (see Section 1):

$$\overline{W}^\lambda_{\mu\nu} = \overline{\Gamma}^\lambda_{\mu\nu} + \frac{1}{3}\delta^\lambda_\mu \overline{W}_\nu \quad (\text{D.1})$$

and

$$\overline{\Gamma}^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + \overline{Q}^\lambda_{\mu\nu} + \Delta^\lambda_{\mu\nu} \quad (\text{D.2})$$

where $\tilde{\Gamma}^\lambda_{\mu\nu}$ is a Levi-Civita connection induced by $g_{(\alpha\beta)}$ on E and

$$\overline{Q}^\nu_{\gamma\mu} = \frac{1}{2} \left(K_{\gamma\mu}{}^\nu - 2g_{[[\mu}{}^\alpha K_{\gamma]\alpha\beta} g^{[\nu\beta]} \right) \quad (\text{D.3})$$

$$\Delta^\nu_{\gamma\mu} = \tilde{g}^{(\nu\delta)} \left\{ K_{\delta(\gamma}{}^\alpha g_{[\mu]\alpha]} + g_{[\rho}{}^\beta \left[g_{([\mu}{}^\rho K_{\gamma)\alpha\beta} g_{\delta]}{}^\alpha - K_{\delta\alpha\beta} g_{([\gamma}{}^\alpha g_{\mu]\rho]} \right] \right\} \quad (\text{D.4})$$

$$g_{\alpha\beta} = g_{(\alpha\beta)} + g_{[\alpha\beta]} \quad (\text{D.5})$$

$$K_{\alpha\beta\gamma} = -\tilde{\nabla}_\alpha g_{[\beta\gamma]} - \tilde{\nabla}_\beta g_{[\gamma\alpha]} + \tilde{\nabla}_\gamma g_{[\alpha\beta]}. \quad (\text{D.6})$$

$\tilde{\nabla}$ is a covariant derivative with respect to the Levi-Civita connection $\tilde{\Gamma}^\alpha_{\beta\gamma}(\tilde{w}^\alpha_\beta)$. We have of course

$$\tilde{g}^{(\alpha\beta)} g_{(\alpha\gamma)} = \delta^\beta_\gamma \quad (\text{D.7})$$

and

$$g_{[\gamma}{}^\beta = \tilde{g}^{(\beta\alpha)} g_{[\gamma\alpha]}. \quad (\text{D.8})$$

Starting from the formula for a 2-form of a curvature:

$$\overline{\Omega}^\alpha_\beta(\overline{W}) = d\overline{W}^\alpha_\beta + \overline{W}^\alpha_\gamma \wedge \overline{W}^\gamma_\beta$$

one gets

$$\overline{\Omega}^\alpha_\beta(\overline{W}) = \tilde{\Omega}^\alpha_\beta - \tilde{\nabla}_{[\delta} \left(\overline{Q}^\alpha_{|\beta|\gamma]} + \Delta^\alpha_{|\beta|\gamma]} \tilde{\theta}^\delta \wedge \tilde{\theta}^\gamma \right) - \frac{2}{3} \delta^\alpha_\beta \overline{W}_{[\delta,\gamma]} \tilde{\theta}^\delta \wedge \tilde{\theta}^\gamma$$

where $\tilde{\Omega}^\alpha_\beta$ is a 2-form of a curvature for a Levi-Civita connection \tilde{w}^α_β on E .

From the formula above we can easily read a tensor of a curvature $\overline{R}^\alpha_{\beta\delta\gamma}(\overline{W})$ and afterwards a Moffat–Ricci tensor as given below:

$$\overline{R}_{\beta\delta}(\overline{W}) = \widetilde{\overline{R}}_{\beta\delta} - \frac{1}{2}(\widetilde{\nabla}_\delta \overline{Q}^\delta_{\beta\gamma} + \widetilde{\nabla}_\delta \widetilde{\Delta}^\delta_{\beta\gamma}) + \frac{1}{4}(\widetilde{\nabla}_\gamma \Delta^\alpha_{\beta\alpha} - \widetilde{\nabla}_\alpha \Delta^\alpha_{\beta\gamma}) \quad (\text{D.9})$$

where $\widetilde{\overline{R}}_{\beta\gamma}$ is a Ricci tensor for a Levi-Civita connection generated by $g_{(\alpha\beta)}$. We get also Moffat–Ricci tensor for \overline{W}^λ_μ . One gets

$$\overline{R}_{\beta\mu}(\overline{W}) = \overline{R}_{\beta\mu}(\overline{W}) + \frac{2}{3}\overline{W}_{[\beta,\mu]}. \quad (\text{D.10})$$

The final result reads

$$\overline{R}_{\beta\mu} = \widetilde{\overline{R}}_{\beta\gamma} - \frac{1}{2}\widetilde{\nabla}_\delta \overline{Q}^\delta_{\beta\gamma} + \frac{1}{4}\widetilde{\nabla}_\gamma \Delta^\alpha_{\beta\alpha} - \frac{3}{4}\widetilde{\nabla}_\delta \Delta^\delta_{\beta\gamma} + \frac{2}{3}\overline{W}_{[\beta,\gamma]}. \quad (\text{D.11})$$

One can also write a scalar curvature

$$\overline{R}(\overline{W}) = g^{(\beta\gamma)}\widetilde{\overline{R}}_{\beta\gamma} - \frac{1}{2}g^{[\beta\gamma]}\widetilde{\nabla}_\delta \overline{Q}^\delta_{\beta\gamma} - \frac{3}{4}g^{(\beta\gamma)}\widetilde{\nabla}_\delta \Delta^\delta_{\beta\gamma} + \frac{1}{4}g^{\beta\gamma}\widetilde{\nabla}_\gamma \Delta^\alpha_{\beta\alpha} + \frac{2}{3}g^{[\beta\gamma]}\overline{W}_{[\beta,\gamma]}. \quad (\text{D.12})$$

Thus now we can write Einstein equations

$$\overline{R}_{\alpha\beta}(\overline{W}) = 8\pi T^{\text{em}}_{\alpha\beta} \quad (\text{D.13})$$

in the following way

$$\widetilde{\overline{R}}_{\beta\gamma} - \frac{1}{2}\widetilde{\nabla}_\delta \overline{Q}^\delta_{\beta\gamma} - \frac{3}{4}\widetilde{\nabla}_\delta \Delta^\delta_{\beta\gamma} + \frac{1}{4}\widetilde{\nabla}_\gamma \Delta^\alpha_{\beta\alpha} + \frac{2}{3}\overline{W}_{[\beta,\gamma]} = 8\pi T^{\text{em}}_{\beta\gamma}. \quad (\text{D.14})$$

Taking symmetric and antisymmetric part of equation (D.14) one gets

$$\widetilde{\overline{R}}_{\beta\gamma} = 8\pi T^{\text{em}}_{(\beta\gamma)} + \frac{3}{4}\widetilde{\nabla}_\delta \Delta^\delta_{\beta\gamma} - \frac{1}{4}\widetilde{\nabla}_{(\gamma} \Delta^\alpha_{\beta)\alpha} \quad (\text{D.15})$$

$$-\frac{1}{2}\widetilde{\nabla}_\delta \overline{Q}^\delta_{\beta\gamma} + \frac{1}{4}\widetilde{\nabla}_{[\gamma} \Delta^\alpha_{\beta]\alpha} + \frac{2}{3}\overline{W}_{[\beta,\gamma]} = 8\pi T^{\text{em}}_{[\beta\gamma]}. \quad (\text{D.16})$$

One can eliminate \overline{W}_μ from the theory using (D.16) and getting

$$\frac{1}{4}\widetilde{\nabla}_{[[\gamma} \Delta^\alpha_{\beta]]\alpha[\mu]} - \frac{1}{2}\widetilde{\nabla}_\delta \overline{Q}^\delta_{[\beta\gamma,\mu]} = 8\pi T^{\text{em}}_{[[\beta\gamma],\mu]}. \quad (\text{D.17})$$

Let us consider our second Maxwell equation, i.e. Eq. (1.46) writing it in a new way. One gets

$$\begin{aligned} \widetilde{\nabla}_\mu F^{\alpha\mu} &= \Delta^\mu_{\delta\mu} F^{\delta\alpha} - \overline{Q}^\alpha_{\delta\mu} F^{\delta\mu} \\ &+ \overline{\nabla}_\mu \left(g^{\alpha\beta} g^{\mu\gamma} \widetilde{g}^{(\tau\rho)} (F_{\rho\gamma} g_{[\beta\tau]} - F_{\rho\beta} g_{[\gamma\tau]}) \right) + 2g^{[\alpha\beta]} \overline{\Delta}_\mu (g^{[\nu\beta]} F_{\nu\beta}). \end{aligned} \quad (\text{D.18})$$

In this way we get Einstein equations Eq. (D.15) and second pair of Maxwell equations in GR (D.18).

Moreover, we get a supplementary condition Eq. (D.17). In this way our unified theory is equivalent to GR plus additional “matter fields”. Moreover, our “matter field” has pure geometrical origin. We get additional terms on the right-hand side of Einstein equations (some additional

terms for an effective energy-momentum tensor). We get also Eq. (1.30) (i.e. $g^{[\mu\nu]}_{,\nu} = 0$) which is a field equation for skewon $g_{[\mu\nu]}$. We get also additional currents on the right-hand side of the second pair of Maxwell equations. This is similar to the case of Einstein–Cartan theory (see Ref. [88] and references cited therein). In this case the theory is described by a metric (symmetric) tensor and a metric connection on a space-time, which can have non-zero torsion. The external sources are energy-momentum tensor (not necessary symmetric) and spin density. One gets the following equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi t_{\mu\nu} \quad (\text{D.19})$$

$$Q^\rho_{\mu\nu} + \delta^\rho_\mu - \delta^\rho_\nu Q^\sigma_{\mu\sigma} = 8\pi s^\rho_{\mu\nu}. \quad (\text{D.20})$$

$Q^\rho_{\mu\nu}$ is a tensor of torsion for a metric connection, $R_{\mu\nu}$ and R are Ricci tensor and a scalar curvature for a connection, $t_{\mu\nu}$ is an energy-momentum tensor, $s^\rho_{\mu\nu}$ is a spin density tensor. Equation (D.20) can be solved getting a torsion (and a contorsion)

$$Q^\rho_{\mu\nu} = 8\pi(s^\rho_{\mu\nu} + \frac{1}{2}\delta^\rho_\mu s^\sigma_{\nu\sigma} + \frac{1}{2}\delta^\rho_\nu s^\sigma_{\mu\sigma}). \quad (\text{D.21})$$

Finally, we get a connection on a space-time and we can rewrite Eq. (D.19) in the following way:

$$\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R} = 8\pi T^{\text{eff}}_{\mu\nu} \quad (\text{D.22})$$

$$T^{\text{eff}}_{\mu\nu} = T_{\mu\nu} - 2\pi(s_{\mu\nu\gamma}s^\gamma + 2s_{\mu\gamma\delta}s^{\delta\gamma}_\nu + s_{\mu\gamma\delta}s^{\gamma\delta}_\nu) + g_{\mu\nu}(s_\gamma s^\gamma - s_{\delta\gamma\alpha}s^{\alpha\gamma\delta} - \frac{1}{2}s_{\delta\gamma\alpha}s^{\gamma\alpha\delta}) \quad (\text{D.23})$$

$$T_{\mu\nu} = t_{\mu\nu} + \frac{1}{2}\pi\tilde{\nabla}_\rho(s_{\nu\mu}^\rho + s_{\nu}^\rho{}_\mu + s_\mu^\rho{}_\nu) \quad (\text{D.24})$$

$$s_\alpha = s^\gamma_{\alpha\gamma}$$

where $\tilde{R}_{\mu\nu}$, \tilde{R} , $\tilde{\nabla}$ are Ricci tensor, scalar curvature and covariant derivative with respect to the Levi-Civita connection generated by $g_{\alpha\beta}$. Formula (D.24) gives us Belifante–Rosenfeld symmetrization of the canonical energy-momentum tensor. On the right-hand side of Eq. (D.22) we have additional “matter fields”, and on the left-hand side a typical term—Einstein tensor in GR.

We can perform a very similar procedure in the case of Moffat (see Ref. [7]) and Einstein–Cartan–Moffat theory (see Ref. [89]) using both procedures described above. In the case of Kaluza–Klein Theory with torsion (see Refs [35], [36]) a similar to Einstein–Cartan Theory method can be applied. In all of these cases GR with additional “matter fields” uses the same metric tensor as in original theory, which is not true in the case of Refs [82], [83], [84], [85], [86], [87]. Thus we have no problems with the physical interpretation of a symmetric metric and we can consider our theory as GR plus additional “matter fields”, which have a geometrical interpretation. In this way we can complete the Einstein programme of a geometrization of physical interactions getting “interference effects” between gravitational and electromagnetic fields and prove that GR is a distinguished gravitational theory among alternative theories of gravitation and unified field theories.

Acknowledgement

I would like to thank Professor B. Lesyng for the opportunity to carry out computations using MathematicaTM 6¹ in the Centre of Excellence BioExploratorium, Faculty of Physics, University of Warsaw. I would like to thank an anonymous referee for critical comments to improve my paper. The SCOAP3 sponsorship of my paper is highly appreciated.

References

- [1] Kalinowski M. W., *Nonsymmetric Fields Theory and its Applications*, World Scientific, Singapore, New Jersey, London, Hong Kong 1990.
- [2] Kalinowski M. W., *Can we get confinement from extra dimensions*, in: *Physics of Elementary Interactions* (ed. Z. Ajduk, S. Pokorski, A. K. Wróblewski), World Scientific, Singapore, New Jersey, London, Hong Kong 1991.
- [3] Kalinowski M. W., *Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory in a general non-abelian case*, *Int. Journal of Theor. Phys.* **30**, p. 281 (1991).
- [4] Kalinowski M. W., *Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory in the electromagnetic case*, *Int. Journal of Theor. Phys.* **31**, p. 611 (1992).
- [5] Kalinowski M. W., *Scalar fields in the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory*, arXiv: hep-th/0307242v9, 7 May 2004.
- [6] *McGraw–Hill Dictionary of Scientific and Technical Terms*, sixth edition, McGraw–Hill, New York, 1989.
- [7] Moffat J. W., *Generalized theory of gravitation and its physical consequences*, in *Proceeding of the VII International School of Gravitation and Cosmology*. Erice, Sicilly, ed. by V. de Sabbata, World Scientific Publishing Co., Singapore, p. 127, 1982.
- [8] Mann R. B., *New ghost-free extensions of general relativity*, *Classical Quantum and Gravity* **6**, p. 41 (1989).
- [9] Mann R. B., *Five theories of gravity*, *Classical Quantum and Gravity* **1**, p. 561 (1984).
- [10] Qi X.-L., Hughes T. L., Zhang S.-C., *Topological field theory of time-reversal invariant insulators*, *Phys. Rev. B* **78**, p. 195424 (2008).
- [11] Post E. J., *Formal Structure of Electromagnetics (General Covariance and Electromagnetics)*, North Holland, Amsterdam, 1962, and Dover Publications Inc., Meneda, NY, 1998.
- [12] Serdyukov A., Semchenko I., Tretyakov S., Sihvola A., *Electromagnetics of Bi-anisotropic Materials (Theory and Applications)*, Gordon and Breach Science Publ., Amsterdam, 2001.

¹MathematicaTM is the registered mark of Wolfram Co.

- [13] Hehl F. W., Obukhov Yu. N., *Foundations of Classical Electrodynamics (Charge, Flux and Metric)*, Birkhäuser, Basel, 2003.
- [14] Lindell I. V., *Differential Forms in Electromagnetics*, IEEE Press Series on Electromagnetic Wave Theory, Wiley–Blackwell and IEEE Press, 2004.
- [15] Flanders H., *Differential Forms with Applications to the Physical Sciences*, Academic Press, New York, 1963, and Dover, New York, 1989.
- [16] Born M., Infeld L., *Foundation of a new field theory*, Proc. Roy. Soc. London **A144**, p. 425 (1934).
- [17] Plebański J., *Lectures on Nonlinear Electrodynamics*, an extended version of lectures given at Niels Bohr Institute and NORDITA, Copenhagen, 1970.
- [18] Landau L. D., Lifshitz E. M., Pitaevskii L. P., *Electrodynamics of Continuous Media*, v. 8 (Course of Theoretical Physics, second revised edition), Elsevier–Butterworth–Heinemann, 2004.
- [19] de Groot S. R., Suttorp R. G., *Foundations of Electrodynamics*, North Holland Publ. Company, Amsterdam, 1972.
- [20] Antoci S., *Stationary Axially Symmetric Solutions of the Hermitian Theory of Relativity: Generalization of the Papapetrou Class of Vacuum Fields*, Lettere al Nuovo Cimento **36**, p. 16 (1983).
- [21] Ashtekar A., Lewandowski J., *Background quantum gravity: a status report*, Classical and Quantum Gravity **21**, R53 (2004).
- [22] Lal K. B., Ali Najaf, *The wave solutions of the field equations of Einstein's, Bonnor's and Schrödinger's non-symmetric unified field theories in generalized Takeno-space-time*, Tensor (N.S.) **21**, p. 241 (1970).
- [23] Lal K. B., Khare D. C., *On solutions of Einstein's unified field equations of gravitation and electromagnetism in $V_2 \times V_2$ space-time*, Tensor (N.S.) **20**, p. 335 (1969).
- [24] Vaidya P. C., *Unified gravitational and electromagnetic waves*, Progr. Theoret. Phys. **25**, p. 305 (1961).
- [25] Joshi R. L., Husain S. I., *Total radiation in Einstein's unified field theory*, Tensor (N.S.) **15**, p. 66 (1964).
- [26] Takeno H., Ikeda M., Abe S., *On solutions of new field equations of Einstein and those of Schrödinger*, Progr. Theoret. Phys. **6**, p. 837 (1951).
- [27] Lorentz H. A., *Weiterbildung der Maxwell'schen Theorie: Elektronentheorie*, Enzyklopädie der mathematischen Wissenschaften, Band V2, Heft 1, Art. 14, p. 145 (1904).
- [28] Abraham M., *Theorie der Elektrizität II*, Teubner, Leipzig, 1905, 1923.

- [29] Rohrlich F., *Classical Charged Particles*, Addison Wesley, Radwood City, CA, 1990.
- [30] Spohn H., *Dynamics of Charged Particles and their Radiation Field*, Cambridge University Press, Cambridge, 2004.
- [31] Born M., Infeld L., *Electromagnetic mass*, Nature **132**, p. 970 (1933).
- [32] Gambini R., Pullin J., *The loop quantum gravity black hole*, arXiv: 1302.5265v1[gr-qc], 21 February 2013.
- [33] Schouten J. A., *Der Ricci-Kalkül*, Verlag von Julius Springer, Berlin, 1924.
- [34] Schouten J. A., *Ricci-Calculus*, Springer, Berlin, 1954.
- [35] Kalinowski M. W., *Gauge fields with torsion* (in Polish), Ph.D. thesis submitted to the Faculty of Physics of University of Warsaw, June 1978; published in full in English: Internat. J. Theoret. Phys. **20**, p. 563 (1981).
- [36] Kalinowski M. W., *Torsion and the Kaluza–Klein Theory*, Acta Phys. Austriaca **57**, p. 45 (1985).
- [37] Thirring W., *Fivedimensional theories and CP-violation*, Acta Phys. Austriaca Suppl. IX, p. 256, Springer, Wien–New York, 1972.
- [38] Kalinowski M. W., *CP-nonconservation and a dipole electric moment of fermion in the Klein–Kaluza Theory*, Acta Phys. Austriaca **53**, p. 229 (1981).
- [39] Kalinowski M. W., *On a dipole electric moment of fermion in the Kaluza–Klein Theory* (talk), in: Grand Unified Theories and Related Topics (Kyoto 1981), World Scientific, Singapore, p. 452, 1981.
- [40] Kawai T., *A five dimensional unification of the Vierbein and electromagnetic fields*, Progress of Theoretical Physics **67**, p. 1946 (1982).
- [41] Kawai T., *A five dimensional unification of the Vierbein and electromagnetic fields II*, Progress of Theoretical Physics **68**, p. 1365 (1982).
- [42] Kalinowski M. W., $\frac{3}{2}$ -spinor field in the Kaluza–Klein Theory, Acta Phys. Austriaca **55**, p. 167 (1983).
- [43] Kalinowski M. W., *Minimal coupling scheme for Dirac’s field in the nonsymmetric theory of gravitation*, Int. J. Mod. Phys. **A1**, p. 227 (1986).
- [44] Kalinowski M. W., *CP nonconservation and electric dipole moment of fermions in the Nonsymmetric Kaluza–Klein Theory*, Int. J. Theor. Phys. **26**, p. 21 (1987).
- [45] Rayski J., *Unified theory and modern physics*, Acta Physica Polonica **31**, p. 87 (1965).
- [46] Beringer J. et al. (Particle Data Group), *2012 review of particle physics*, Phys. Rev. **D86**, 010001 (2012).

- [47] Booth M. J., *The electric dipole moment of W and electron in the standard model*, arXiv: hep-ph/9301293v3, 4 Feb 1993.
- [48] Jarlskog C., *Commutator of the quark mass matrices in the standard electroweak model and a measure of maximal CP nonconservation*, Phys. Rev. Lett. **55**, p. 1039 (1985).
- [49] Raidal et al., *Flavor physics of leptons and dipole moments*, Eur. Phys. J. **C57**, p. 13 (2008).
- [50] Heckel B. R., *Results from a search for an electric dipole moment of ^{199}Hg* , Physics Procedia **17**, p. 92 (2011).
- [51] Baron J. et al. (The ACME Collaboration), *Order of magnitude smaller limit on the electric dipole moment of the electron*, arXiv: 1310.7534v2 [physics.atom-ph], 7 Nov 2013.
- [52] Pospelova M., Ritz A., *CKM benchmarks for electron EDM experiments*, arXiv: 1311.5537v2 [hep-ph], 28 Nov 2013.
- [53] Giunti C., Kim C. W., *Fundamentals of Neutrino Physics and Astrophysics*, Oxford Univ. Press, Oxford, 2007. Errata: 30 April 2013.
- [54] Dvornikov M., *Field theory description of neutrino oscillations*, arXiv: 1011.4300v2 [hep-ph], 6 Apr 2011.
- [55] Dvornikov M., *Neutrino spin-flavor oscillations in electromagnetic fields of various configurations*, arXiv: 0708.3572v1 [hep-ph], 27 Aug 2007.
- [56] Kalinowski M. W., *Vanishing of the cosmological constant in non-Abelian Kaluza–Klein Theories*, International Journal of Theoretical Physics **22**, p. 385 (1983).
- [57] Trautman A., *Fibre bundles associated with space-time*, Rep. Mathematical Physics **1**, p. 29 (1970/71).
- [58] Einstein A., *The Meaning of Relativity*, Appendix II, Fifth Edition, revised, Methuen and Co., London, 1951, p. 127.
- [59] Einstein A., *On the generalized theory of gravitation*, Scientific American **182**, p. 13 (1950).
- [60] Hlavatý V., *Geometry of Einstein’s Unified Field Theory*, P. Noordhoff Ltd., Groningen 1957.
- [61] Kalinowski M. W., *The program of geometrization of physics. Some philosophical remarks*, Synthese **77**, p. 129 (1988).
- [62] Klotz A. H., *Macrophysics and Geometry: From Einstein’s Unified Field Theory to Cosmology*, Cambridge Univ. Press, Cambridge, 1982.
- [63] Kalinowski M. W., *The Nonsymmetric Kaluza–Klein Theory*, J. Math. Phys. **24**, p. 1835 (1983).
- [64] Kalinowski M. W., *Material sources in the Nonsymmetric Kaluza–Klein Theory*, J. Math. Phys. **25**, p. 1045 (1984).

- [65] Overduin J. M., Wesson P. S., *Kaluza–Klein Gravity*, Phys. Rep. **283**, p. 303 (1997).
- [66] Janssen T., Prokopec T., *Instabilities in the Nonsymmetric Theory of Gravitation*, Classical and Quantum Gravity **23**, p. 4967 (2006).
- [67] Damour T., Deser S., McCarthy J., *Nonsymmetric gravity theories: Inconsistencies and a cure*, Phys. Rev. D **47**, p. 1541 (1993).
- [68] Trautman A., *On the propagation of information by waves*, in: Recent Developments in General Relativity, Pergamon, Oxford; PWN, Warsaw, 1962.
- [69] Kalinowski M. W., *On the old-new method of solving nonlinear equations*, J. Math. Phys. **25**, p. 2620 (1984).
- [70] Zakharov V. D., *Gravitational waves in Einstein’s Theory* (in Russian), Nauka, Moscow, 1972; English translation: Israel Program for Scientific Translations, Halsted Press, Jerusalem–London.
- [71] Kobayashi S., Nomizu K., *Foundations of Differential Geometry*, vols. I and II, John Wiley & Sons Interscience, New York, 1963 and 1969.
- [72] Wu T. T., Yang C. N., *Concept of nonintegrable factors and global formulation of gauge fields*, Phys. Rev. D **12**, p. 3845 (1975).
- [73] Pokorski S., *Gauge Field Theories*, second edition, Cambridge University Press, Cambridge, 2000.
- [74] Kalinowski M. W., Mann R. B., *Linear approximation in the Nonsymmetric Kaluza–Klein Theory*, Classical and Quantum Gravity **1**, p. 157 (1984).
- [75] Wrede R. C., “*n*” *Dimensional Considerations of Basic Principles A and B of the Unified Theory of Relativity*, Ph.D. Thesis submitted to the Faculty of the Graduate School of Indiana University, August 1956; published partially in Tensor (N.S.) **8**, p. 95 (1958).
- [76] Barut O., Rączka R., *Theory of Group Representations and Applications*, PWN, Warsaw, 1977.
- [77] Atiyah M. F., Bott R., Shapiro A., *Clifford modules*, Topology Supplement **3**, p. 1 (1964).
- [78] Cartan E., *The Theory of Spinors*, Hermann, Paris, 1966.
- [79] Trautman A., *The Dirac operator on hypersurfaces*, Acta Physica Polonica **B26**, p. 1283 (1995).
- [80] Trautman A., *Connections and the Dirac operator on spinor bundles*, Journal of Geometry and Physics **58**, p. 238 (2008).
- [81] Borowiec A., *The Dirac equation on a Kaluza–Klein bundle*, Bulletin of the Polish Academy of Sciences, Ser. Physics **XXVII.2**, p. 65 (1979).

- [82] Ferrans M., Kijowski J., *On the equivalence of the relativistic theories of gravitation*, Gen. Rel. and Gravitation **14**, p. 165 (1982).
- [83] Jakubiec A., Kijowski J., *On the universality of Einstein equations*, Gen. Rel. and Gravitation **19**, p. 719 (1987).
- [84] Jakubiec A., Kijowski J., *On theories of gravitation with nonlinear Lagrangian*, Phys. Rev. **D37**, p. 1406 (1988).
- [85] Jakubiec A., Kijowski J., *On the universality of linear Lagrangians for gravitational field*, J. Math. Phys. **30**, p. 1073 (1989).
- [86] Jakubiec A., Kijowski J., *On theories of gravitation with nonsymmetric connection*, J. Math. Phys. **30**, p. 1077 (1989).
- [87] Kijowski J., Werpachowski R., *Universality of affine formulation in General Relativity*, Reports on Mathematical Physics **59**, p. 1 (2007).
- [88] Trautman A., *Einstein–Cartan theory*, Encyclopedia of Mathematical Physics, ed. J.-P. Francoise, G. L. Naber, T. S. Tsun, Elsevier, Oxford, 2006, vol. 2, p. 189.
- [89] Kalinowski M. W., *An Einstein–Cartan–Moffat theory*, Phys. Rev. **D26**, p. 3419 (1982).